Lesson Plan Teacher Course

Pre-Assessment	
Teaching hours:	flexible
Target group:	(Pre-service) teachers
Description	
Goals:	• Gain overview of (pre-service) teacher knowledge prior to the participation in this course
Structure:	• Select the tasks that are of interest for the planned course • The same tasks should be used at the end of the course. Additional feedback questions can be added.

Tasks

Activity 1.

Have you ever encountered the term functional thinking? YES / NO

If so, what do you consider functional thinking to be? If not, what is your guess? Expand your answer.

Activity 2.

From what age, in your opinion, is it possible to develop functional thinking as was explained higher? Justify your answer.

Activity 3.

What topics in the mathematical curriculum (and in which grades) are relevant for functional thinking development? Explain.

Activity 4.

What do you consider to be a goal of teaching about functions? Expand your answer.

Activity 5.

How do you know if students attain the goals you formulated in the previous question? Design three tasks that are good to assess students' functional thinking. Solve them, specify the age (grade) for which they are aimed, and justify your selection.

Activity 6.

What learning difficulties and misconceptions do you expect when teaching functions? Use as many examples as possible to depict your answer.

Activity 7.

At a sushi restaurant, *nigiri* costs 4.50 € per serve and *sashimi* costs 9.00 € per serve. Hiroko spent a total of 45 € buying *x* serves of *nigiri* and *y* serves of *sashimi*.

- **A.** Find a function formula describing the relation between the number of *nigiri* serves (x) and the number of *sashimi* serves (y) bought by Hiroko.
- **B.** Sketch the graph of the function found in (A)

Activity 8.

Decide, whether the functions below are linear. Justify your reasoning.

A. $y = 14$

a. It is linear function. b. It is not linear function.

Explanation:

B. $x = 14$

a. It is linear function. b. It is not linear function.

Explanation:

C. $y = 2x + 1$

a. It is linear function. b. It is not linear function.

Explanation:

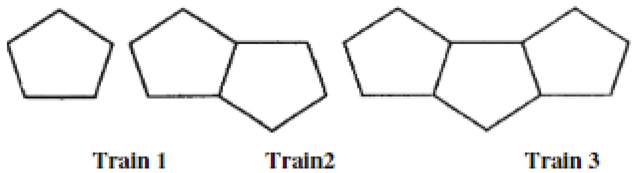
D. $y = \frac{x^2}{x} + 1$

a. It is linear function. b. It is not linear function.

Explanation:

Activity 9.

The first three trains – consisting of equilateral congruent pentagons – in the pattern are shown below:

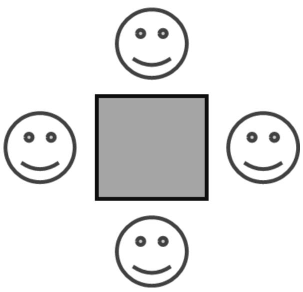


A) Determine the perimeter for the 4^{th} train.
B) Determine the perimeter for the 100^{th} train.
C) Write a description that could be used to find the perimeter of any train in the pattern. Explain how you know. How does your description relate to the visual representation of the trains?

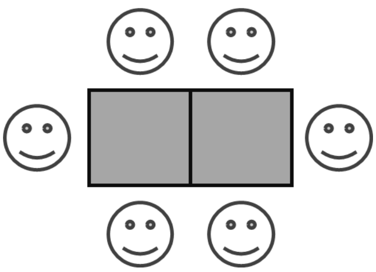
Activity 10.

Brady is having his friends over for a birthday party. He wants to make sure he has a seat for everyone. He has square tables.

He can seat 4 people at one square table in the following way:



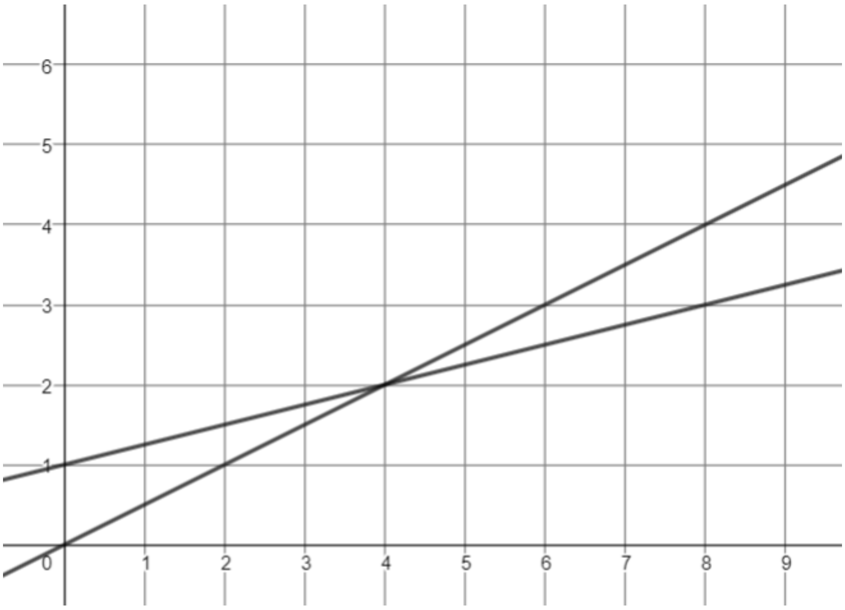
If he joins another square table to the first one, he can seat 6 people:



a) If Brady has 8 tables, how many people can he seat as his birthday party? And how about 20 tables? Explain your answer.
b) Suggest three different questions based on the pattern related to the seating task that could be administered to students. Explain the purpose of each question.

Activity 11.

Study the following graphs.



Suggest three different questions based on the graphs that could be administered to students. Explain the purpose of each question.

Activity 12.

Patterns and linear functions are both subjects in the mathematics curriculum. Which didactical relation do you see between the two and how do you foster this relation with your students? Please explain.

Activity 13.

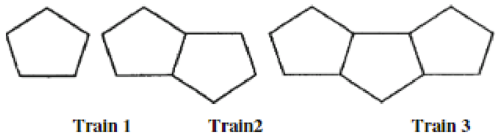
State the definition of the following concepts:

A) function:
B) linear function:

Activity 14.

1. Discuss **mathematical** problems which can be related to the following tasks:

<table>
<tr>
<td>The first three trains – consisting of equilateral congruent pentagons – in the pattern are shown below:



2. Determine the perimeter for the 4^{th} train.
3. Determine the perimeter for the 100^{th} train.
4. Write a description that could be used to find the perimeter of any train in the pattern.
Explain how you know.
How does your description relate to the visual representation of the trains?</td>
<td>From the following table, determine the formula of the function:
<table><tr><td>x</td><td>0</td><td>1</td><td>2</td><td>3</td></tr><tr><td>f(x)</td><td>3</td><td>5</td><td>7</td><td>9</td></tr></table></td>
</tr>
<tr>
<td>**Possible problems:**</td>
<td>**Possible problems:**</td>
</tr>
</table>

Activity 15.

Design a lesson plan to introduce the topic of linear function for grade 9.

Your task is to prepare the lesson. When writing the preparation, imagine that, according to it, someone else should teach and quickly understand what it is all about, or that you want to return to this preparation in a few years and you don't want to spend time thinking about how each task is solved, why you actually put it there, etc.

Therefore, include solutions to tasks, comments on tasks that seem important to you from the teacher's point of view, etc. You can insert photographed solutions into the preparation, or scan it. Just keep the homepage structure as shown in the template.

Work out this preparation to the best of your ability. Feel free to use whatever is available to you BUT be sure to cite it. Please work independently and use the attached template.

The subject of the preparation is the Definition of a linear function - learning new subject matter.

Please name the document that you send back Surname_DDMa_LF

At the end of the preparation (or even in between) you can write notes if you are not sure about something.

Feedback Questionnaire

Please rate the following statements in context of the teacher training course.

(1 – Absolutely Disagree / 5 – Absolutely Agree):

	1	2	3	4	5
1. I learned interesting things in the teacher training course about mathematics.	○	○	○	○	○
2. I learned interesting things in the teacher training course about teaching.	○	○	○	○	○
3. The knowledge I gained is useful for teaching Mathematics.	○	○	○	○	○
4. The structure of the teacher training course was appropriate and effective.	○	○	○	○	○
5. The content of the teacher training course was appropriate and effective.	○	○	○	○	○
6. The way of delivery of the teacher training course was appropriate and effective.	○	○	○	○	○
7. I will use the material of the project in my teaching.	○	○	○	○	○
8. I will use the digital tools of the project in my teaching.	○	○	○	○	○
9. The digital material of the project is interesting.	○	○	○	○	○
10. The digital material of the project facilitates conceptual understanding of the mathematics concepts.	○	○	○	○	○
11. I would recommend this seminar to a colleague of mine.	○	○	○	○	○

Please answer the following questions.

1. What did you like best about this teacher training course?
2. What suggestions do you have for improving this teacher training course?
3. What other suggestions would you like to share with us regarding this teacher training course?

Thanks for completing this questionnaire!

Lesson Plan Teacher Course

Knowledge of Topic of Functions	
Teaching hours:	180 minutes
Target group:	Pre-service teachers
Description	
Goals:	• Pre-service teachers know how to solve tasks connected to functions. • They understand the own knowledge concerning the functions and are aware of possible gaps. • They are aware of different types of task connected with functions.
Structure:	• 90 minutes Individual tasks solving including self-assessment • 75 minutes Discussion about the tasks • 15 minutes Self-reflection

Activities

Tasks solving and the discussion (90 + 75 minutes)

Students solve the problems independently, they can also solve them at home. In addition to solving the problems, they also fill in a self-assessment card to work within the following discussion. The aim of the self-assessment is, among other things, to create a space to look critically at their mathematical knowledge.

We then analyse the individual tasks in a joint meeting. Suggestions for discussion are given below the tasks.

Activity 1. Lift

The hotel has several floors, the ground floor with number zero, and below the ground floor there are several parking floors. The following table shows which floor you can reach after a few seconds. You have checked in on the 14th floor and are about to take the lift down to the parking lot to your car.

Number of seconds	Number of floors
0	14
2	10
4	6
6	2
7	?

A. Which floor will the lift be on after seven seconds? Explain your answer.

B. At what speed is the lift descending? Explain your answer.

The correct solution:

a)

Počet sekúnd	Počet poschodí
0	14
2	10
4	6
6	2
7	?

4 = op
4 = op
4 = op

(a) Na ktorom poschodí bude výťah po siedmich sekundách? Vysvetlite svoju odpoveď.

na prízemí, pretože každé 2 sekundy klesne o 4 poschodia, o 1 s klesne o 2 poschodia

(a) Na ktorom poschodí bude výťah po siedmich sekundách? Vysvetlite svoju odpoveď.

za 2 sekundy prejde 4 poschodia ... za 1 sekundu 2 poschodia:
teda za 7 sekúnd ... 7.2 = 14
za 7 sekúnd prejde výťah 14 poschodí
→ teda bude na prízemí

b)

(b) Akou rýchlosťou klesá výťah? Vysvetlite svoju odpoveď.

Výťah klesá rýchlosťou 2 poschodia za sekundu

Rýchlosť vypočítame ako $v = \frac{s}{t} = \frac{\text{dráha (poschodia)}}{\text{čas}}$

z tabuľky jasne vyplýva, že výťah klesne za 2 sekundy o 4 poschodia, teda z toho určíme rýchlosť 4 poschodia / 2 sek = = 2 poschodia za sekundu.

Possible problems:

s = vt

poítame rýchlos pre 10. poschodie:
naše s = koko poschodí
naše v = neznáma
naše t = 2 sekundy (výah sa po 2s dostane na 10 poschodie)

doplníme do vzorca: 4 = v . 2 → 1 = v . 0,5
2 = v   2m/s = v
2 m/s = v

riešenie: výah sa pohybuje rýchlosou 2 m/s.

- erroneous thinking about units

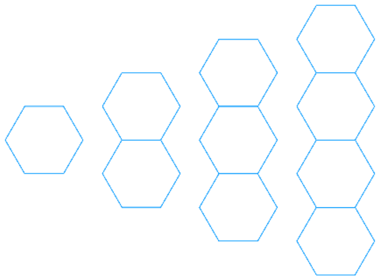
Activity 2. Hexagons

The first figure (1 hexagon) has a perimeter of 6.

The third figure (3 hexagons on top of each other) has a perimeter of 14.

The second figure (2 hexagons on top of each other) has perimeter ____________.

The fifth figure (5 hexagons on top of each other) has perimeter ____________.



A. Describe how you would determine the perimeter of a figure composed of 100 hexagons on top of each other without knowing the perimeter of a figure composed of 99 hexagons on top of each other.

B. Write a formula to calculate the perimeter for any number of hexagons in a chain above.

C. Explain why your formula should be correct.

Different formulas - Discuss with students how each formula came about.

- $o = 6 + (6 - 2)(n - 1)$
- $o = 6n - 2(n - 1)$
- $o = 5.2 + 4n$

Expected problems in solving:

(b) Napíšte vzorec na výpočet obvodu pre ľubovoľný počet šesťuholníkov v reťazci nad sebou.

O = obvod útvaru v reazci
n = poet 6-uhoníkov v reazci
o = obvod jedného šesuholníka
s = poet spolených strán šesuholníkov v reazci

O = (n . o) - 2s

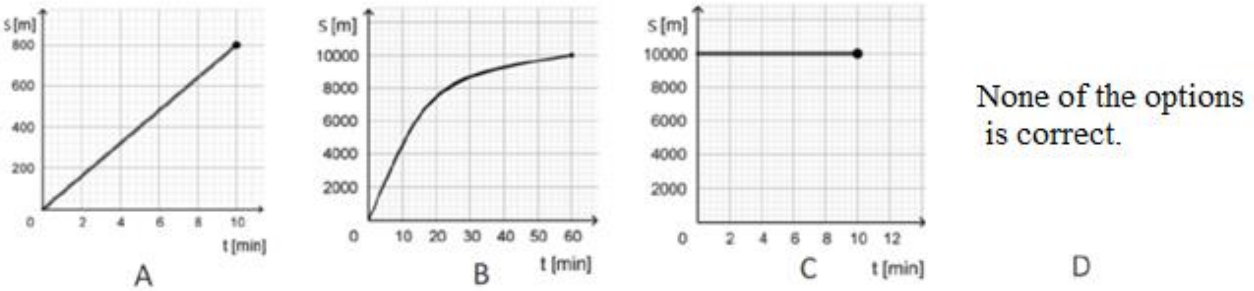
(c) Vysvetlite prečo by mal byť váš vzorec správny.

Poda ma je môj vzorec jednoducho pochopitený a jasný.

Activity 3. Adam is running

Adam started from point *A* and ran 10 km (see picture).

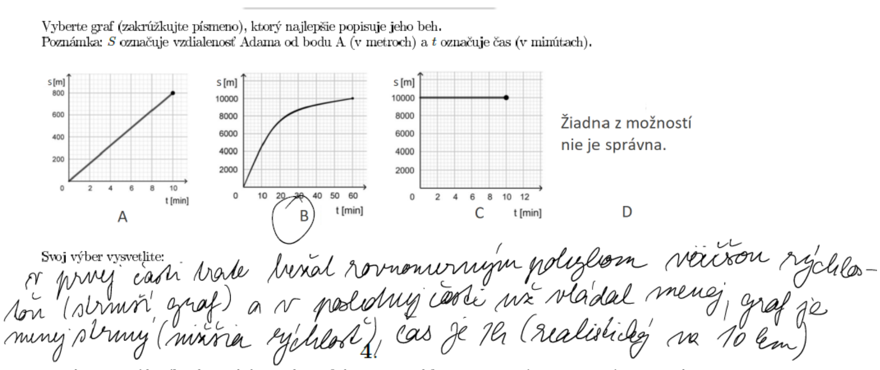
Choose a graph (circle the letter) that can describe its run and justify your choice.



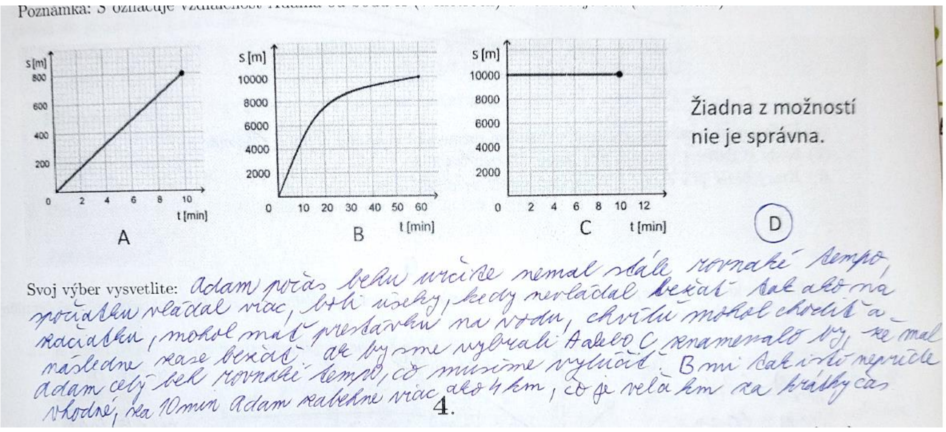
Note: S denotes the distance of Adam from point *A* (in meters) and *t* denotes the time (in minutes).

Correct solution - open a discussion between B and D

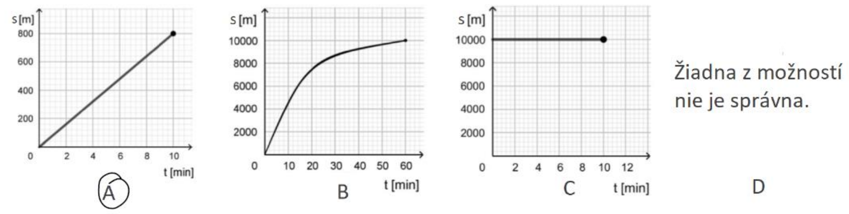
(B)



(D) We also accept D if there is an argument regarding an unrealistically fast run in the first half of the time.



Possible problems:



Svoj výber vysvetlite:
C to nemôže byť pretože by od 0 minúty mal prejdetých 10 km
môže to byť B al. A
B- vyjadruje že na začiatku bežal rýchlo a potom spomalil
A - vyjadruje že bežal konšt. rých. 4.

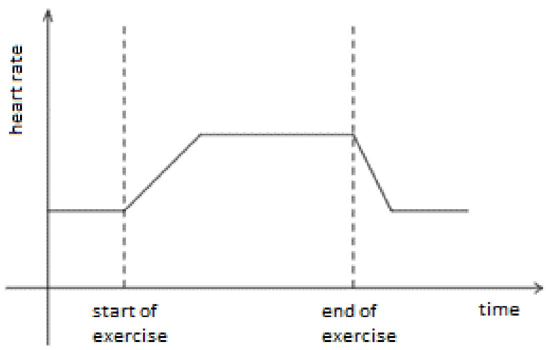
- Problem with the necessity of linear motion - not solving for values on axes

Activity 4. Without training

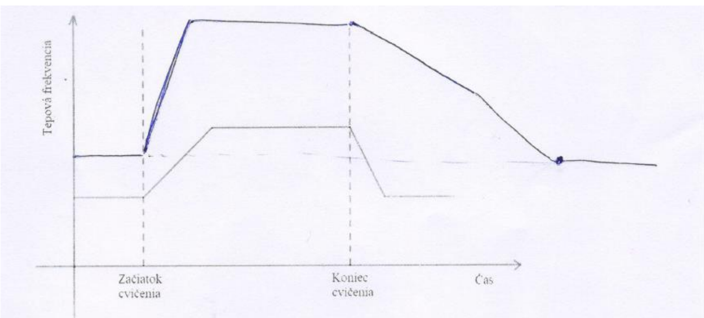
The textbook states, "There are several differences between the heart rate waveform of a regularly exercising - trained and untrained person:

- The trained person has a lower resting heart rate before the start of the exercise,
- Her heart rate rises more slowly with exercise and reaches lower values,
- Her heart rate drops faster after exercise and returns to resting value in a shorter time."

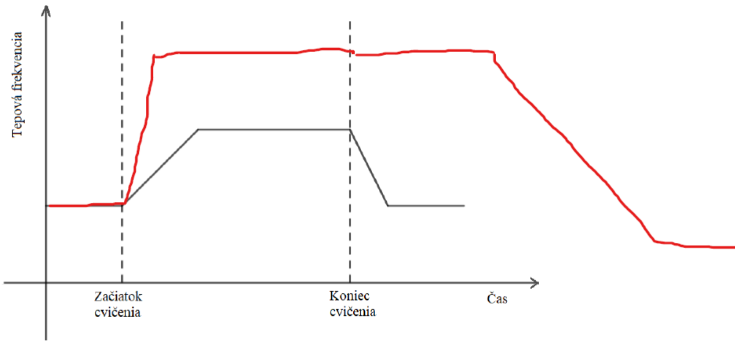
In the figure is a graph of the heart rate of a trained person. In the same figure, sketch what the heart rate graph of an untrained person would look like for the same exercise, satisfying all the above differences.



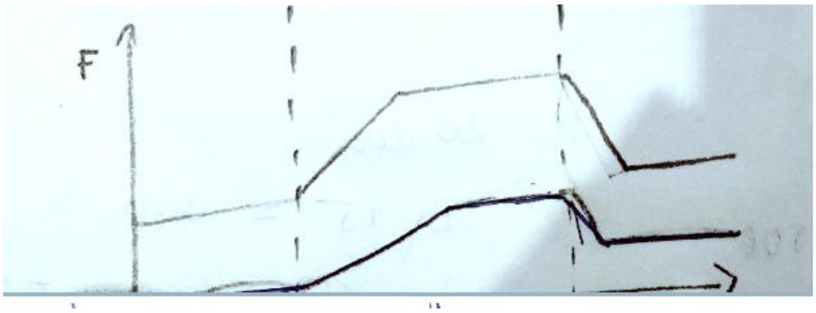
The correct solution



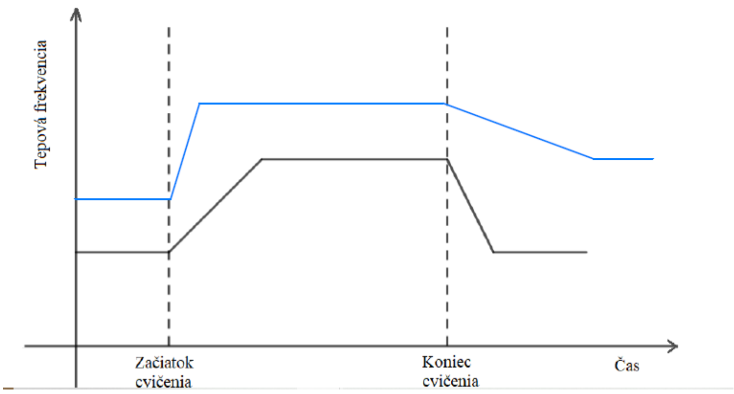
Faulty reasoning - In addition to discussing the following solutions, we can give a problem: What would the problem have to sound like for these graphs to be correct?



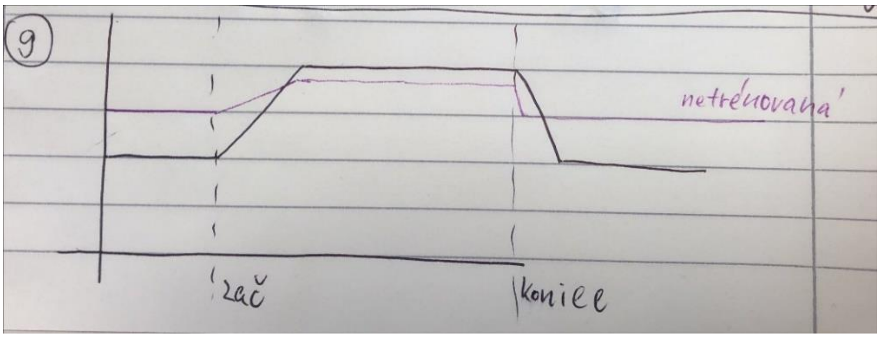
- An untrained person would have a lower resting heart rate with each additional exercise



- does not respect faster acceleration and slower descent



- An untrained person would have a higher resting heart rate with each additional exercise



- The problem with the Directive

Activity 5. Runners

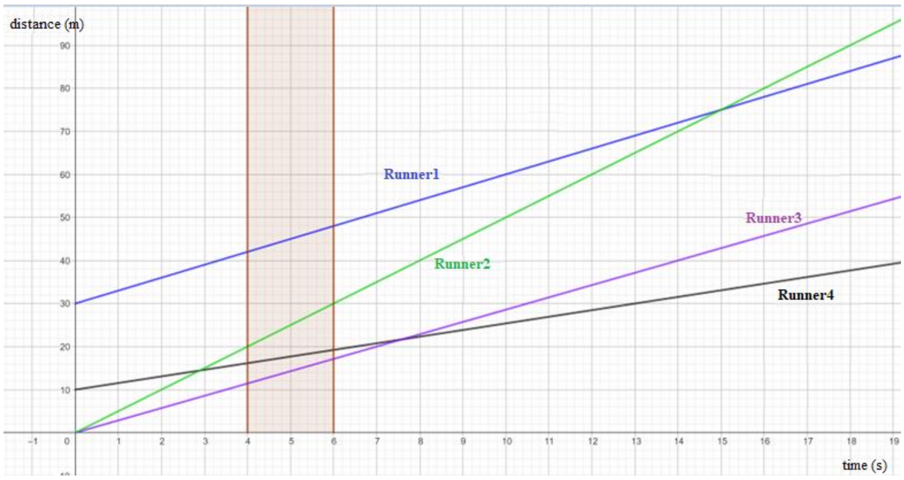
Find out the following information from the figure:

A. How many meters will **Runner 2** run in the time range t = 4s to t = 6s?

B. When is **Runner 1** faster than **Runner 2**?

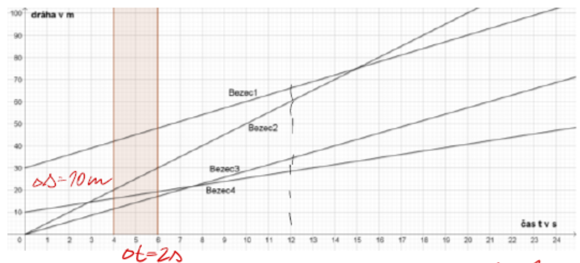
C. Which runner is the fastest at time t = 12s?

Give reasons for your answers.



The correct solution:

Z nasledujúceho obrázku zistite



(a) Koľko metrov zabehne *Bežec 2* v časovom rozmedzí $t = 4s$ až $t = 6s$? $v_2 = \frac{\Delta s}{\Delta t} = \frac{10m}{2s} = 5m/s$

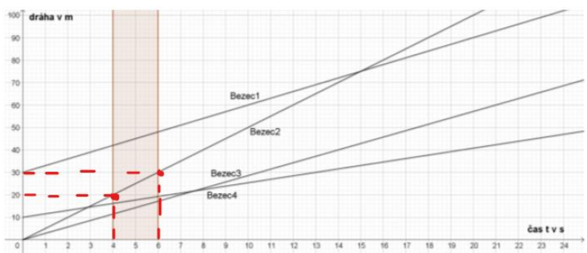
(b) Kedy je *Bežec 1* rýchlejší ako *Bežec 2*? nikdy (graf B1 je vždy menej strmý ako B2 → vždy je pomalší)

(c) Ktorý bežec je v čase $t = 12s$ najrýchlejší?

bežec 2 → najstrmší graf

Flawed reasoning - how would the graph have to be changed to make it true that runner 1 was fastest from 1st to 15th second? What would the interpretation be then? Point out the inconsistency between the answer to a) and b).

Z nasledujúceho obrázku zistite



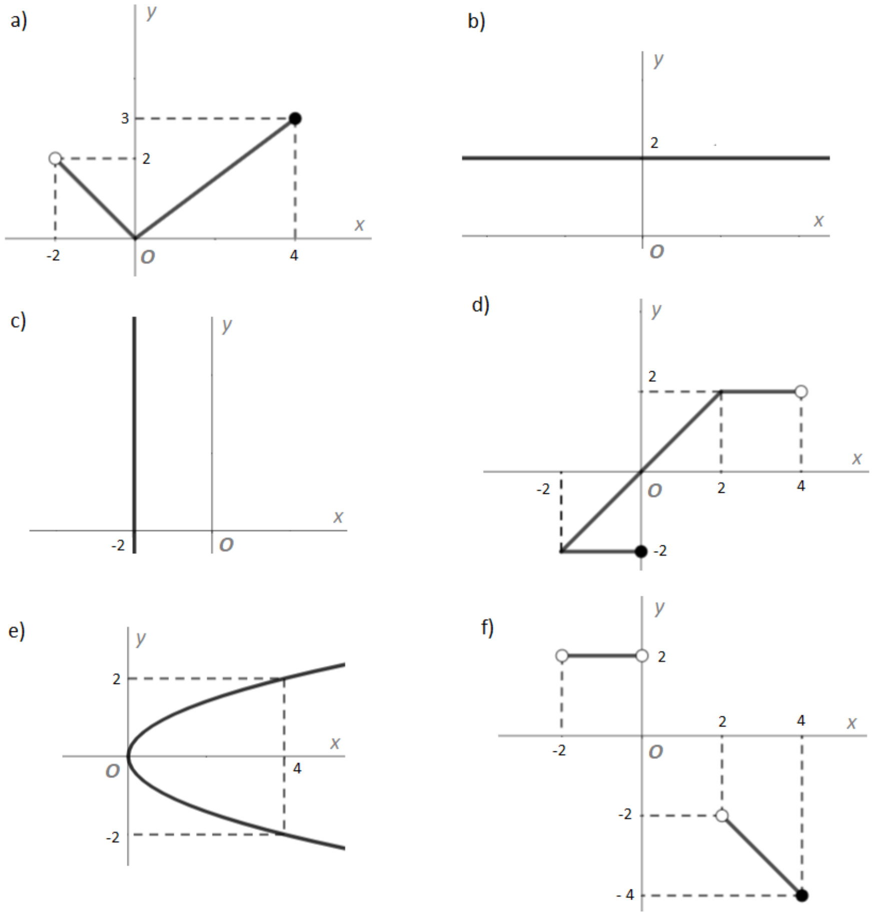
(a) Koľko metrov zabehne *Bežec 2* v časovom rozmedzí $t = 4s$ až $t = 6s$? 10 metrov

(b) Kedy je *Bežec 1* rýchlejší ako *Bežec 2*? od zaiatku, bežec 2 ho dobehne 15 sekunde

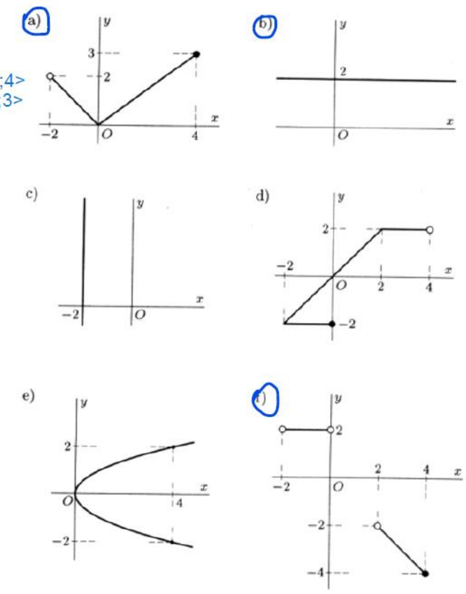
(c) Ktorý bežec je v čase $t = 12s$ najrýchlejší? bežec 1

Activity 6. Graph of a function

Decide which of the graphs shown in the following figures is the graph of a function. If possible, determine the defining domain and the range of the function.



The correct solution:



D(a)=(-2;4>
H(a)=<0;3>

D(b)=R
H(b)={2}

D(f)= (-2;0)U(2;4>
H(f)= <-4,-2)U{2}

**Activity 7.	Formula from the table**

Continue to fill in the table and find the formula of a function:

x	1	2	100
$f(x)$	5	8	

Faulty reasoning - Outline a coordinate system an the points in it [1,5] a [2,8] - what graph of the function can pass through these points?

7.

Pokračujte vo vypĺňaní tabuľky a nájdite analytický predpis funkcie.

x	1	2	100
$f(x)$	5	8	302

z grafu funkcie vidíme, že ide o lineárnu funkciu, kt. predpis je y = ax + b, do tohto vzorca dosadíme čísla z tabuky:

$5 = a \cdot 1 + b$
$5 = a + b$
$5 - a = b$

$8 = 2a + b$
$8 - 2a = b$

$5 - a = 8 - 2a$
$a = 3$

$5 = 1 \cdot 3 + b$
$2 = b$

$y = 3x + 2$

The correct solution:

7/ úloha má aj veľa riešení - my vieme nájsť aj veľa funkcií, ktoré prechádzajú zadanými bodmi a sú rozdielne vo všetkých ostatných.
Ak budeme uvažovať, že f(x) je lineárna, potom:

Dd mal

x	1	2	3	4	5	6	7	100
$f(x)$	5	8	11	14				

$\Rightarrow f(x) = 3x + 2$

Potom

x	1	2	3	4	5	6	7	100
$f(x)$	5	8	11	14	17	20	23	302

7.

Pokračujte vo vypĺňaní tabuľky a nájdite analytický predpis funkcie.

x	1	2	3	4	5	6	7	100
$f(x)$	5	8	11	14	17	20	23	302

$f(x) = 5 + 3(x-1)$

$f(100) = 5 + 3 \cdot 99 = 5 + 297 = 302$

8.

Activity 8. Verbal description of a function

Decide whether the relations described below are functions. Circle your answer and give reasons for your decision:

A. Dependence of theme park admission price on age as shown in the table:

age (years)	Price (€)
0 - 2	0
3 - 15	15
16+	25

Yes / No, Reason:

B. The area of the triangle ABC, which is assigned to the length of its side AB.

Yes / No, Reason:

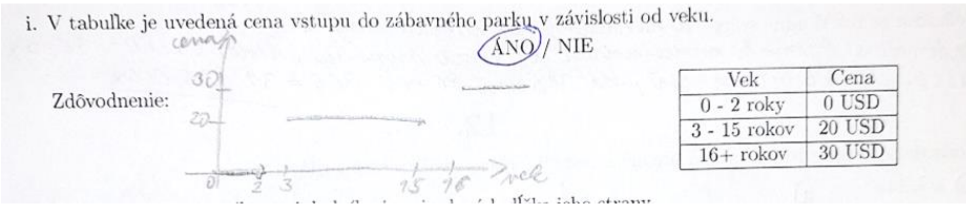
C. To the number of pages in a mathematics textbook is assigned the number of sentences on that page.

Yes / No, Reason:

D. To a student in your class are assigned the names of their siblings.

Yes / No, Reason:

A. Draw a graph


i. V tabuľke je uvedená cena vstupu do zábavného parku v závislosti od veku.

ÁNO / NIE

Zdôvodnenie:

Vek	Cena
0 - 2 roky	0 USD
3 - 15 rokov	20 USD
16+ rokov	30 USD

Are we OK with this solution?

i. V tabuľke je uvedená cena vstupu do zábavného parku v závislosti od veku.

ÁNO / NIE

Zdôvodnenie: je to priama úmernosť rastúca funkcia

Vek	Cena
0 - 2 roky	0 USD
3 - 15 rokov	20 USD
16+ rokov	30 USD

b) How would the assignment have to be changed to be a function? If we add the word "equilateral" to the assignment - what would the prescription look like?

ii. Plocha rovnostranného trojuholníka je priradená k dĺžke jeho strany.

ÁNO / NIE

Zdôvodnenie:

$$S(a) = a \cdot a \cdot \sin 60^0 = a^2 \cdot \frac{\sqrt{3}}{2}$$

ii. Plocha rovnostranného trojuholníka je priradená k dĺžke jeho strany.

ÁNO / NIE

Zdôvodnenie:

(dĺžka) strana	1	2	3	...
plocha	$\frac{1 \cdot \sqrt{3}}{2}$	$\frac{2\sqrt{3}}{2}$	$\frac{3\sqrt{3}}{2}$	

c) What feature is the student talking about in the incorrect solution?

ÁNO / NIE

Zdôvodnenie:

Každá strana má práve jeden pocet viet (nemôže byt na jednej strane aj 5 aj 6 viet)

ÁNO / NIE

Zdôvodnenie: môže byť viac strán s rovnakým počtom viet a preto by neplatilo, že každému x je priradné práve 1 y.

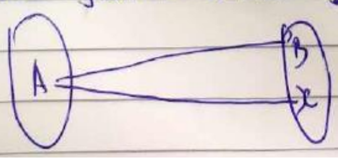
d) What is the definition of a function? Does it have to be an assignment between numerical sets?

iv. Žiakovi vo vašej triede sú priradené mená jeho súrodencov.

ÁNO / NIE

Zdôvodnenie: 1.) nemáme len číselné hodnoty
2.) Jednému menu / osobe priradíme aj viac rovnakých hodnôt.

d) Nie - jednému žiakovi vieme priradiť 2 a viac mien



iv. Žiakovi vo vašej triede sú priradené mená jeho súrodencov.

ÁNO / NIE

Zdôvodnenie:

AK SÚ PRIRADENÉ AKO TROJICA {A, 0, C} teda jeden prvok ÁNO ak nie... tak nie je to FUNKCIA

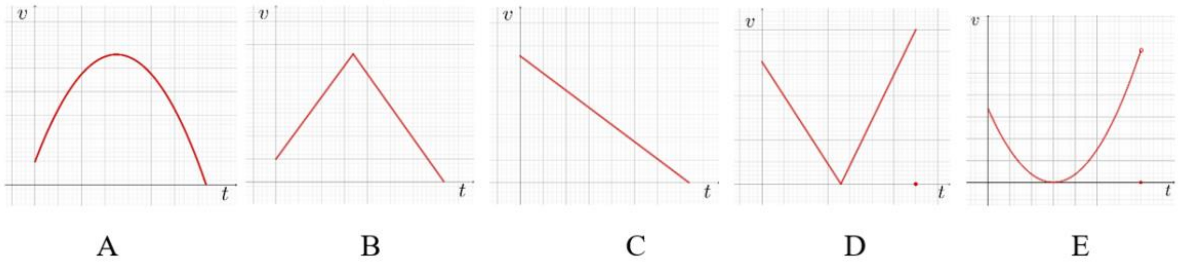
iv. Žiakovi vo vašej triede sú priradené mená jeho súrodencov.

ÁNO / NIE

Zdôvodnenie: v triede môžu byť dvaja súrodenci ktorí majú ďalšieho spoločného súrodenca

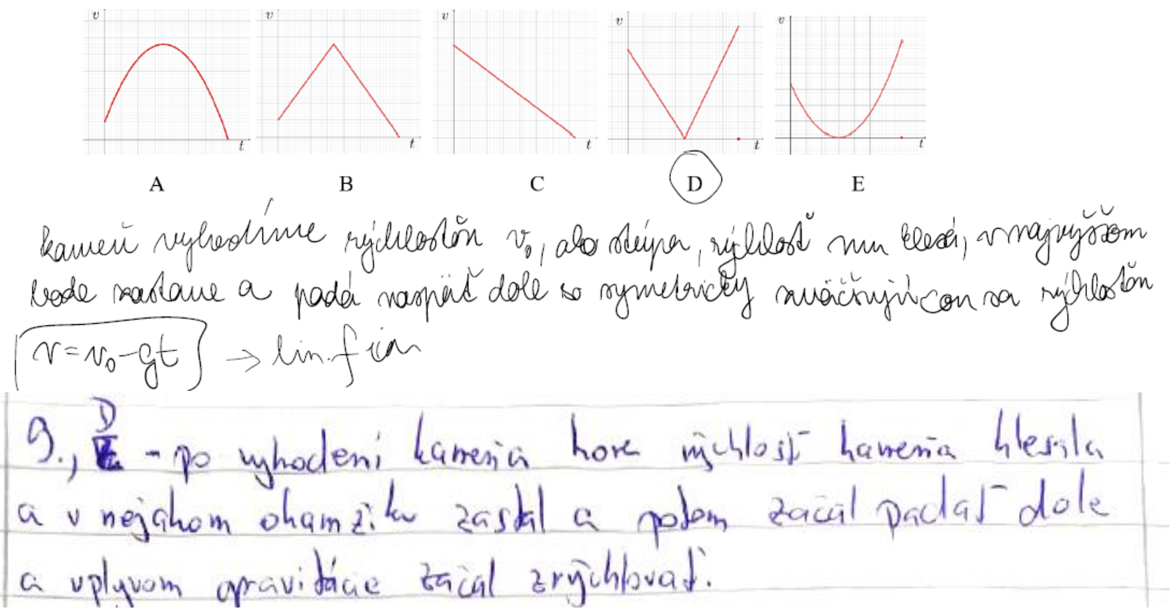
Activity 9.   Stone

A stone was thrown vertically upwards. Select one of the graphs that illustrate the correspondence between velocity and time (neglecting air resistance). Justify your choice.

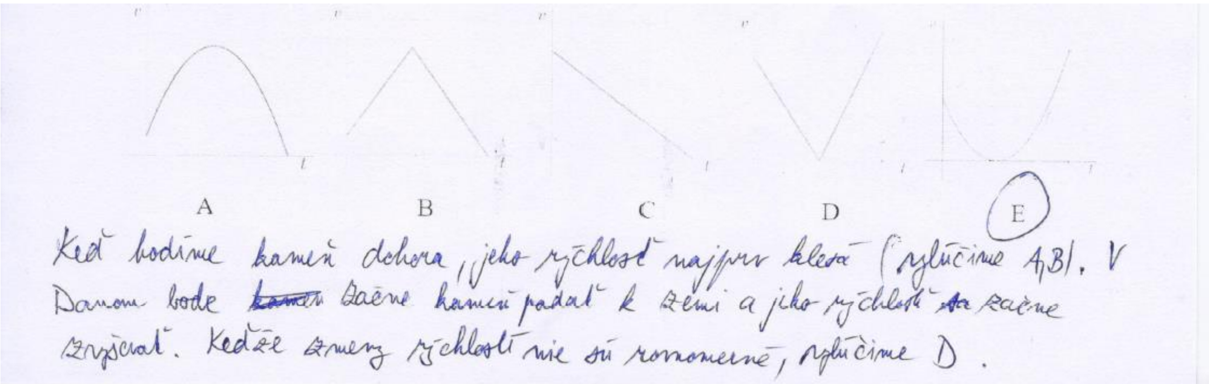


Task: Draw a graph of distance from the ground versus time for each graph. Then discuss the derivation.

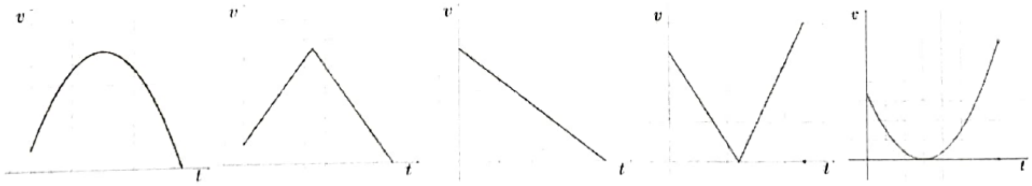
The correct solution:



Faulty reasoning:

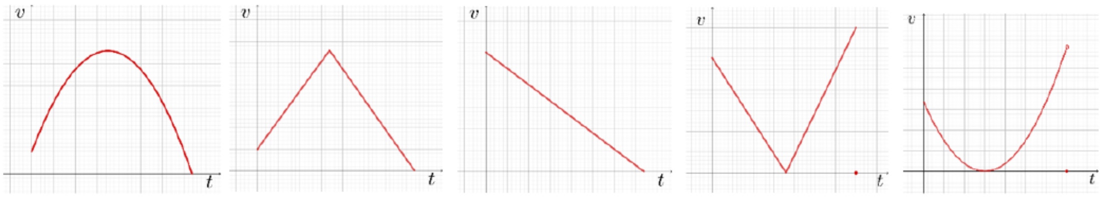


- Velocity varies linearly, and trajectory varies quadratically - a discussion of the differences between D and E can lead to a discussion of derivative. In addition, we may be given the task of sketching graphs for the dependence of velocity or trajectory on time for uniform motion.



A B C D E

kameň ide hore a ďalej odonňa, potom začne šikmo klesať



A B C D E

v pri hode je menšia ako pri páde
t pri hode je ,,väcsi" ako pri páde
kame padá rýchlejšie za kratší as

- misconception "graph as picture" - point out the seriousness of this error

Activity 10. Playground

Helenka wants to organize a birthday party in the children's playroom. She decides between the following offers:

Playground A: The price for each guest is 15 €. 15 per guest.

Playground B: The price for each guest is 12 €. In addition, a fixed cost of 50 € is payable.

Playground C: The price for each guest is 18 €. A discount of 30 € will be given on the final price.

Which offer is the most advantageous for her?

Discussion: Correct solution vs. Incomplete solution - what are good functions for? A short discussion on modeling, on solving equations with parameter

10.) $f(x) = 15x$ $g(x) = 12x + 50$ $h(x) = 18x - 30$

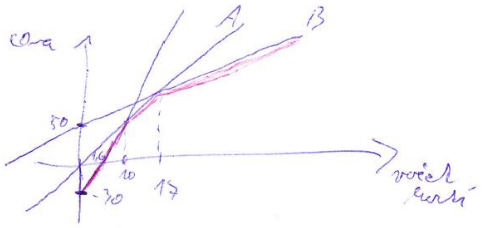
x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
f(x)	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	
g(x)	62	74	86	98	110	122	134	146	158	170	182	194	206	218	230	242	254	
h(x)	-12	6	24	42	60	78	96	114	132	150	168	186	204	222	240	258	276	

Do 10 ľudí je najlepšie objednať ihrisko C
od 11 do 16 ľudí je najlepšie objednať A
Od 17 vyššie je najlepšie objednať B

$A(x) = 15x$

$B(x) = 12x + 50$

$C(x) = 18x - 30$



Ideálne miesto = { C ; pre 0 - 10 hostí
A ; pre 10 - 17 hostí
B ; pre 17 a viac hostí.

Incomplete solutions

• ihrisko C: Cena za každého hosťa je 18 €. Z konečnej cenovej ...

Ktorá ponuka je pre ňu najvýhodnejšia? Závisí to od počtu hostí.

napr. ak je hostí 20, potom:
A: $15 \cdot 20 = 300$
B: $12 \cdot 20 + 50 = 290$ ⇒ najvýhodnejšie
C: $18 \cdot 20 - 30 = 330$

ak je hostí 16:
A: $15 \cdot 16 = 240$ ⇒ najvýhodnejšie
B: $12 \cdot 16 + 50 = 242$
C: $18 \cdot 16 - 30 = 258$

11.

Ktorá ponuka je pre ňu najvýhodnejšia? Nevieme to jednoznačne určiť, závisí to od počtu detí.

Ktorá ponuka je pre ňu najvýhodnejšia? a: y=15x , b: y=12x+50 , c: y=18x-30

závisí od poctu hostí

Activity 11. Population

The table shows Nevada's population from 2000 to 2006.

Year	Population (in millions)	Population growth (millions)
2000	2,020	
2001	2,093	0,073
2002	2,168	0,075
2003	2,246	0,078
2004	2,327	0,081
2005	2,411	0,084
2006	2,498	0,087

A. Divide the population in each year by the population in the previous year. What do you observe?

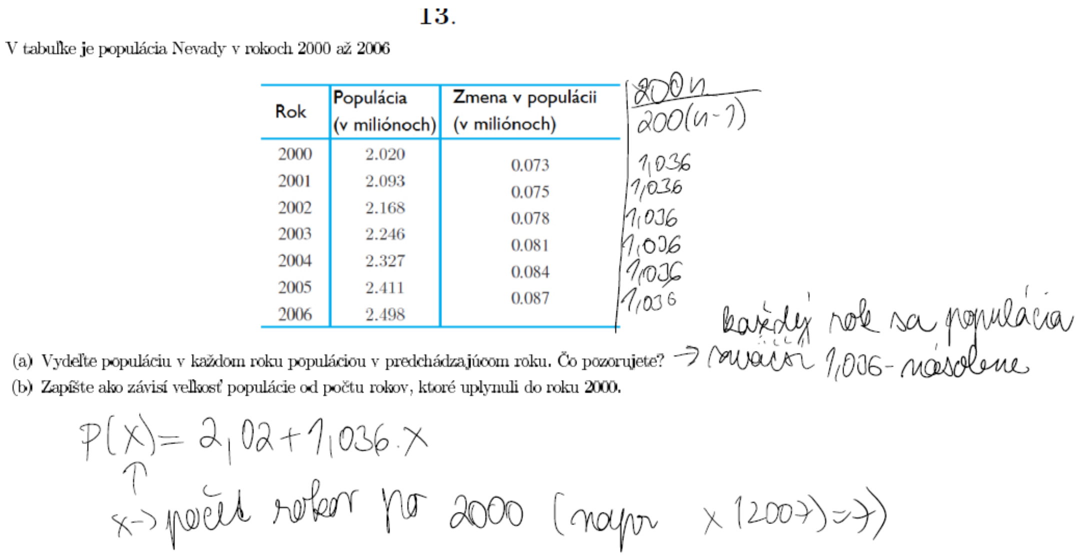
B. Write down how the size of the population depends on the number of years that have passed since 2000.

In case there are many linear functions to be let modeled in the spreadsheet.

The correct solution:

- the ratio of two consecutive terms is about 1.035 - this satisfies the definition of a geometric sequence, or we can see that it is an exponential function.
- $p(x) = 2{,}02 \cdot 1{,}035^x$, where x is the number of years that have elapsed since 2000.

Faulty reasoning:



13.

V tabuľke je populácia Nevady v rokoch 2000 až 2006

Rok	Populácia (v miliónoch)	Zmena v populácii (v miliónoch)	$\frac{200n}{200(n-1)}$
2000	2.020		
2001	2.093	0.073	1,036
2002	2.168	0.075	1,036
2003	2.246	0.078	1,036
2004	2.327	0.081	1,036
2005	2.411	0.084	1,036
2006	2.498	0.087	1,036

(a) Vydeľte populáciu v každom roku populáciou v predchádzajúcom roku. Čo pozorujete? → každý rok sa populácia zväčší 1,036-násobne

(b) Zapíšte ako závisí veľkosť populácie od počtu rokov, ktoré uplynuli do roku 2000.

$P(x) = 2{,}02 + 1{,}036 \cdot x$

x → počet rokov po 2000 (napr. x(2007) = 7)

V tabuľke je populácia Nevady v rokoch 2000 až 2006

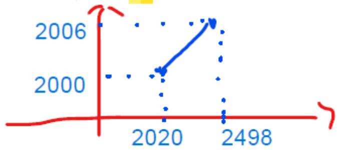
	Rok	Populácia (v miliónoch)	Zmena v populácii (v miliónoch)
	2000	2.020	0.073
1,0361	2001	2.093	0.075
1,0358	2002	2.168	0.078
1,0359	2003	2.246	0.081
1,0361	2004	2.327	0.084
1,0361	2005	2.411	0.087
1,0361	2006	2.498	

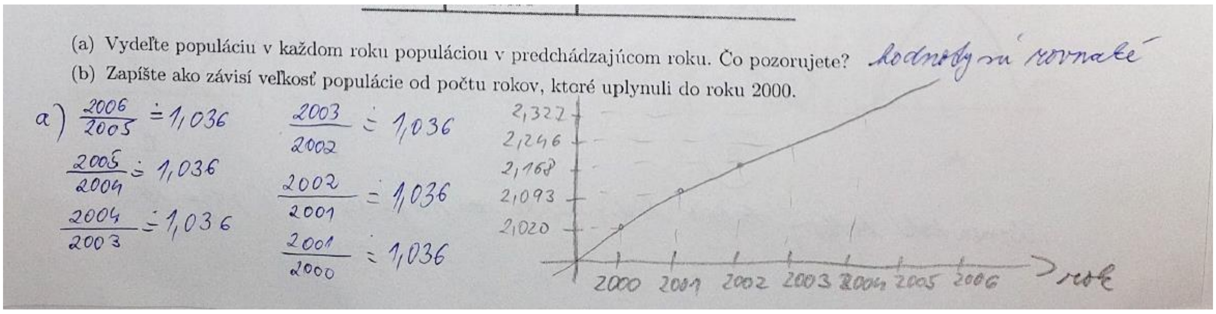
(a) Vydeľte populáciu v každom roku populáciou v predchádzajúcom roku. Čo pozorujete?

Nárast populácie v percentách, hodnoty sú približne rovnaké

(b) Zapíšte ako závisí veľkosť populácie od počtu rokov, ktoré uplynuli do ? roku 2000.

an=an-1*1,036




(a) Vydeľte populáciu v každom roku populáciou v predchádzajúcom roku. Čo pozorujete? hodnoty sú rovnaké

(b) Zapíšte ako závisí veľkosť populácie od počtu rokov, ktoré uplynuli do roku 2000.

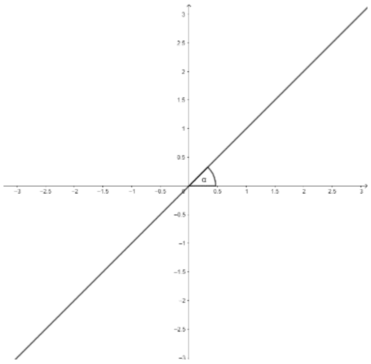
a) $\frac{2006}{2005} \doteq 1,036$ $\frac{2003}{2002} \doteq 1,036$

$\frac{2005}{2004} \doteq 1,036$ $\frac{2002}{2001} \doteq 1,036$

$\frac{2004}{2003} \doteq 1,036$ $\frac{2001}{2000} \doteq 1,036$

Activity 12. Slope

A. The graph represents the function f such that $f\colon \longrightarrow x$.



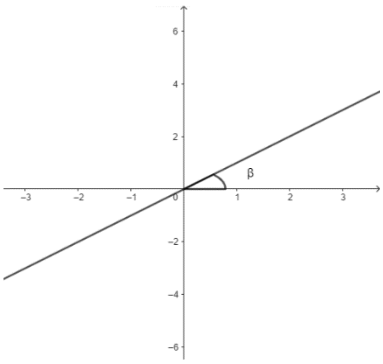
I. What is the function slope? How did you determine it?

II. Divides the graph of a function f halves the angle bisected by the positive part of the coordinate axis x with the positive part of the coordinate axis y? Yes / No

Justification:

III. Can you calculate the tangent of the angle that the graph of a function f with the positive part of the coordinate axis x? If yes, determine its value. If not - because of what?

B. The student used software to draw the same function f. It gave him the following graph:



I. What is the function slope? How did you determine it?

II. Divides the graph of a function f halves the angle bisected by the positive part of the coordinate axis x with the positive part of the coordinate axis y? Yes / No

Justification:

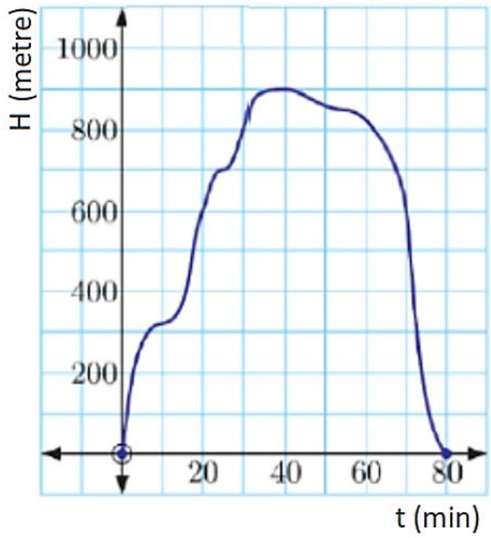
III. Can you calculate the tangent of the angle that the graph of a function f with the positive part of the coordinate axis x? If yes, determine its value. If not - because of what?

Describe your thoughts, reactions, dilemmas, and any ideas you have regarding these issues:

Discussion on the directive and its visual representation, the change of scale.

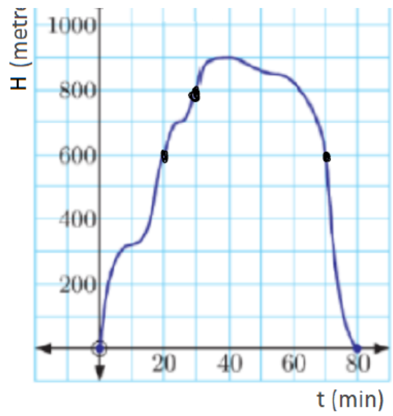
Activity 13. Balloon

When riding in a hot air balloon, the function indicates H the height of the balloon after t minutes of ride. Its graph is shown below:



a) Determine the value $H(30)$ and explain your answer in the context of a hot air balloon ride.
b) Find values of t such that $H(t) = 600$. Explain your answer in the context of a hot air balloon ride.
c) What range of heights was recorded for the balloon?
d) How long was the balloon ride?
e) Can you read the distance travelled by the balloon from the graph of function H ?
Yes / No
Justification:

The correct solution:



$H(30) = 800$

teplovzdušný balón dosiahne za 30 min výšku 800m.

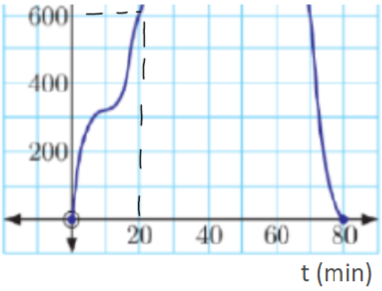
$t = 20$ a 70
v týchto časoch dosahuje výšku 600 m

(a) Určte hodnotu $H(30)$ a svoju odpoveď vysvetlite v kontexte letu teplovzdušným balónom.
(b) Nájdite také hodnoty t, aby $H(t) = 600$. Vysvetlite svoju odpoveď v kontexte letu teplovzdušným balónom.
(c) Aký rozsah výšok bol zaznamenaný pre balón? → 0 - 900 m
(d) Ako dlho trvala jazda balónom? → 80 min
(e) Dokážete z grafu funkcie H odčítať vzdialenosť, ktorú balón prekonal? Zakrúžkujte ÁNO alebo NIE a zdôvodnite svoju odpoveď.

ÁNO / NIE

Zdôvodnenie: Na základe údajov ktoré máme nevieme

Faulty reasoning: a task for students - comment on this solution.



(a) Určte hodnotu $H(30)$ a svoju odpoveď vysvetlite v kontexte letu teplovzdušným balónom. 700 za 30 min sa balón dostal do výšky 700 m

(b) Nájdite také hodnoty t, aby $H(t) = 600$. Vysvetlite svoju odpoveď v kontexte letu teplovzdušným balónom. 20 za 20 min mal nadmorskú výšku 600 m

(c) Aký rozsah výšok bol zaznamenaný pre balón? 0 – 900

(d) Ako dlho trvala jazda balónom? 80 minút

(e) Dokážete z grafu funkcie H odčítať vzdialenosť, ktorú balón prekonal? Zakrúžkujte ÁNO alebo NIE a zdôvodnite svoju odpoveď.

ÁNO / NIE

Zdôvodnenie: neviem ako ale dá sa

Activity 14. Features of the function

A. Is there a function whose defining domain is $(0,5)$ a the range of values is $\langle 2,5\rangle$? If yes, draw its graph, write its prescription, or describe it in some other way.

B. Is there a function whose defining domain is the set of numbers $\{1,2,3\}$ and the doma of values is the set $\{1,2\}$? If yes, draw its graph, write its prescription or describe it in some other way.

C. Is there a function whose defining domain is the set of numbers $\{1,2\}$ and the doma of values is the set $\{1,2,3\}$? If yes, draw its graph, write its prescription, or describe it in some other way.

D. Is there a function that for any real numbers x satisfies the following requirements? If so, draw its graph or describe it in some other way.

$$f(x+y) = f(x) \cdot f(y)$$

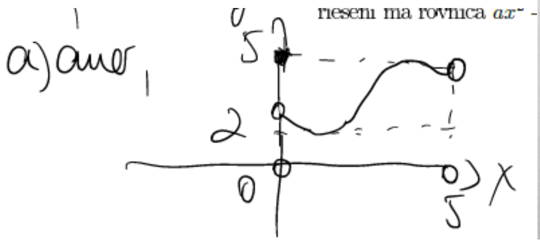
$$f(x+y) = f(x) + f(y)$$

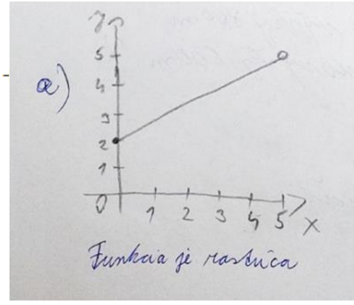
$$f(x \cdot y) = f(x) + f(y)$$

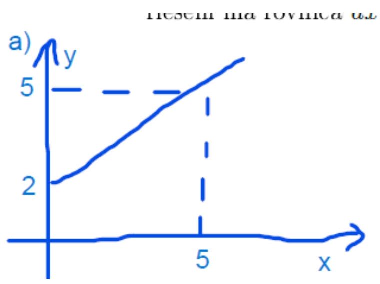
E. If we substitute 1 for x in the expression $ax^2 + bx + c$. we get a positive number, substituting 6 we get a negative number. How many solutions does the equation $ax^2 + bx + c = 0$ have? Justify your answer.

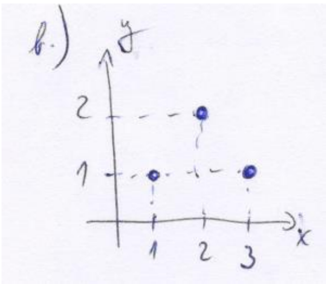
In the discussion we can assess the correctness of the following solutions:

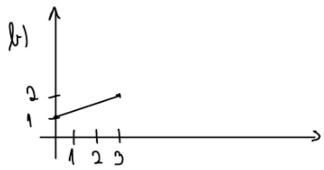
A.









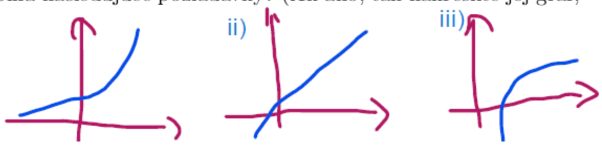


(c) Existuje funkcia, ktorá pre akékoľvek reálne čísla x, y spĺňa nasledujúce požiadavky? (Ak áno, tak nakreslite jej graf, zapíšte jej predpis alebo ju popíšte inak.)

i. $f(x+y) = f(x) \cdot f(y)$ $y=e^x$

ii. $f(x+y) = f(x) + f(y)$ $y=x$

iii. $f(x \cdot y) = f(x) + f(y)$)? $y=\log x$



y) $e^{x+y} = e^x \cdot e^y \Rightarrow f(x) = e^x$

ii) $f(x) = 2x$

$2(x+y) = 2x + 2y$

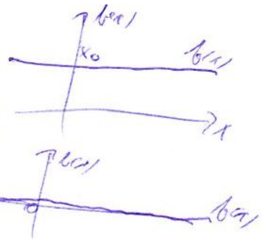
i) $f(x) = \ln x$

$\ln(x \cdot y) = \ln x + \ln y$

c/ i) konštantná funkcia $f(x) = x_0$; ~~x₀∈ℝ~~ $x_0 \in \{1;0\}$

ii) konštantná nulová fcia $f(x) = 0$, $x \in \mathbb{R}$

iii) analogicky ii)



Self-reflection (15 minutes)

In this section, students have space to return to their self-assessment sheet. It is important that they can verbalize what surprised them. This form of feedback can help them take responsibility for their learning. Together we can summarise which task they had the most difficulty with.

Lesson Plan Teacher Course

Function definition & representations	
Teaching hours:	2 x 45 minutes
Target group:	Pre-service teachers
Important links	
Videos:	https://www.funthink.eu/learning-environments/lower-secondary-education/nomogram-intro-and-graph
Description	
Goals:	Pre-service teachers: • reflect on the concept of function and its different definitions, • decide and justify whether the given definitions of a function (and also a linear function) are correct, • discover that, depending on the learning level, a set of ordered pairs can be both the representation of a function (secondary school) and its definition (university level), • recall about the role of representations, • practice to move between different representations of function and choose the best representation depending on the context, • discuss chosen students' misconceptions about representations of functions and the notion of function.
Structure:	• Individual work – solving the 4 tasks and reflection in the whole group and in pairs on **correctness of the solutions, mistakes and misconceptions and false convictions** revealed during the process of solving the tasks, aim of the tasks, different representations • Teaser video on Nomogram learning environment presentation • Representations vs. the notion of function • Listing and analysing the representations • What is a function – different definitions to be assessed

Activities

Representations vs. the notion of function (individual work with Handout 1 & joint reflection 30 minutes)

Activity 1. Individual solving Tasks 1-4 on choosing the proper representation

Pre-service teachers individually solve the following ***TASKS 1, 2, 3, 4 (Handout 1)*** *on choosing the best representations and understanding its role.*

Solve the following four tasks. Reflect on the method of solving and on the aim of each task.

TASK 1. Does a function exist which domain is (0,5) and the set of values is [2,5]? Justify your answer.

TASK 2. Does a function exist which domain is {1,2,3} and the set of values is {1,2}? Justify your answer.

TASK 3. Does a function exist which for any real x, y numbers fulfil the requirement: $f(x + y) = f(x) \cdot f(y)$. Justify your answer.

[or $f(x + y) = f(x) + f(y)$ or $f(x \cdot y) = f(x) + f(y)$]

TASK 4. If we substitute 1 for x in the expression $ax^2 +bx +c$ we get a positive number, substituting 6 we get a negative number. How many solutions does the equation $ax^2 +bx +c =0$ have? Justify your answer.

Activity 2. Joint reflection on the correctness of Tasks 1-4 solutions

Pre-service teachers present their solutions on the board and discuss together with the lecturer ***the correctness of their solutions, mistakes and misconceptions and false convictions*** *revealed during the process of solving the tasks.*

Additional question:

- What is a proper way of justification? How can we answer the questions in Tasks 1-3 "Does the function exist?"?

Response: If the answer is positive (and all of them are) - the simplest way is to give any kind of an example in any representation (any designations) fulfilling the requirement or theoretically prove its existence.

If the answer would be negative (this could be a first approach to Task 1) - it would have to be proven theoretically.

Representations vs. the notion of function
(work in pairs & joint initial discussion – 30 minutes)

Activity 3. Further reflection - discussion in pairs on Tasks 1-4

Pre-service teachers work in groups (pairs) and answer the 4 questions listed below.

If certain themes spontaneously emerge during joint discussion (Activity 2), we either explore them together at that time and do not give them to the pair discussion (Activity 3) or we only partially address them during joint discussion and ask participants to systematise their conclusions in group work.

1. What is a goal of solving Tasks 1-4? What do they explore?
2. What misconceptions might students reveal while solving the tasks? [based also on pre-service teachers' mistakes and false convictions revealed during the process of solving the tasks]
3. What representations of the function appeared in the task solutions? List the different representations and give examples.
4. What are the strengths and weaknesses of the different representations?

Collect conclusions of the first joint discussion and make lists of the representations.

Representations vs. the notion of function
(individual work & work in pairs (Handout 2) & joint initial discussion – 15 minutes)

Activity 4. What is a function? Write your own definition.

Every pre-service teacher formulates anonymously on a small sheet of paper his/her own definition and gives it anonymously to the lecturer for later discussion.

Activity 5. What did you discover?
Discussion in pairs on Handout 2: two definitions of function (history) & representations listed in a textbook

The lecturer distributes to the students Handout 2 which contains two different general definitions of function at university level (1):

1

Source: Sajka (2019, p. 15-16):

“In mathematics, there are two fundamentally different approaches to defining the concept of function, two contrasting definitions of the notion of function formulated by Peano (1911) and Hausdorff (1914):

1. According to Peano (1911), a relation is a certain set of ordered pairs, and in turn a function is a certain special type of relation in which if the pair (x, y) and (x, z) are pairs belonging to the relation, then y = z.

2. Hausdorff (1914) first defined the product of any sets A, B as the set of all ordered pairs p = (a, b), where a ϵ A and b ϵ B, and then wrote:

- *(...) we shall consider a certain set P of such pairs, having, namely, the property that each element* ***a*** *of A occurs in the first place in one and only one pair p of P. Each element* ***a*** *thus determines one and only one element* ***b****, the very one with which in the pair p = (****a****,* ***b****) it is connected; this element determined by* ***a****, dependent on* ***a****, assigned to* ***a****, we denote by b = f(****a****) and we say that by this in A (i.e. for all elements from A) a certain unambiguous function has been defined. We regard two such functions f(****a****), f′(****a****) as equal if and only if the associated sets of pairs P, P' are equal, so that for every* ***a*** *there is f(****a****) = f′(****a****) (Hausdorff, 1914, p. 33)*

Source:

- Hausdorff, F. 1914*, Grundzüge der Mengenlehre*, Leipzig.
- Peano, G. (1911). Sulla definizione di funzione, *Atti della Reale Accademia dei Lincei, Serie 5a, Classe di scienze fisiche, matematiche e naturali* 20, 3-5.
- Sajka, M. (2019*). Pojęcie funkcji. Wiedza przedmiotowa nauczyciela matematyki*. Wydawnictwo Naukowe Uniwersytetu Pedagogicznego, DOI 10.24917/9788380841048.

Handout 2 contains also textbook scans:

The excerpt (2) from a textbook for secondary schools lists different kinds of representations including *set of ordered pairs* - see introduction in page 286, point (d).

English translation of the Textbook and task book scans: (Kurczab, Kurczab, Świda, 2015, 2014):

2

286 *8. Function and its properties*

Ways of describing functions

The most common ways we use to describe functions are:
(a) verbal description
(b) a table
(c) a graph
(d) a set of ordered pairs
(e) a formula
(f) a graph

3

8.10. Dana jest funkcja, przedstawiona w postaci zbioru par uporządkowanych. Narysuj wykres tej funkcji.

a) $\{(-2, -1), (-1, 0), (0, 1), (1, 2), (2, 3), (3, 4)\}$
b) $\{(-4, 3), (-3, 4), (-2, 0), (-1, 1), (0, 3), (2, 4)\}$
c) $\{(-3, 1), (-2, 1), (-1, 1), (0, 1), (1, 1), (2, 1)\}$
d) $\{(-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9)\}$

A function is given, represented as a set of ordered pairs. Draw the graph of this function.

8.11. Dana jest funkcja, opisana za pomocą zbioru par uporządkowanych. Podaj wzór tej funkcji.

a) $\{(-2, 3), (-1, 4), (0, 5), (1, 6)\}$
b) $\{(-4, 8), (-3, 6), (1, -2), (2, -4), (3, -6), (4, -8)$
c) $\left\{\left(-\frac{1}{4}, -\frac{1}{64}\right), \left(-\frac{1}{3}, -\frac{1}{27}\right), \left(-\frac{1}{2}, -\frac{1}{8}\right), (0, 0), (2, 8), (3, 27), (4, 64)\right\}$
d) $\left\{(-8, 5), (-3, 5), \left(-\frac{1}{2}, 5\right), (3, 5), (11, 5)\right\}$

A function is given, represented as a set of ordered pairs. Give the formula of this function.

8.12. Dana jest funkcja, opisana za pomocą zbioru par uporządkowanych:
$\{(1, 3), (-2, 7), (7, 4), (0, 0), (8, 1)\}$.

a) Podaj wartość funkcji dla argumentu 7.
b) Podaj argument funkcji, dla którego wartość funkcji wynosi 1.

A function is given, represented as a set of ordered pairs.
a) Calculate the value for the argument 7.
b) Calculate the argument for which the value of the function equals 1

(...) Draw the graph of the function

8.14. Dana jest funkcja, opisana za pomocą zbioru par uporządkowanych. Narysuj wykres tej funkcji.

a) $\{(x, y): |x + 3| \geq 1 \text{ i } y = -x\}$
b) $\{(x, y): |x - 2| < 2 \text{ i } y = x - 1\}$

Source:

(2) Kurczab, M., Kurczab, E., Świda E. (2015). Textbook: „Mathematics 1 to secondary and technical schools", Wydanie IV, Warszawa, p. 286,

(3) Kurczab, M., Kurczab, E., Świda E. (2014)., Matematyka 1, Zbiór zadań do liceów i techników, [Mathematics 1 - task book to secondary and technical schools], Wydanie III, Warszawa, 2014, p.203.

What is a function? What is a representation?
Common discussion on Handout 2 & Discussion on teachers own definitions (15 minutes)

Activity 6. Let us list the representations of the notion of function

We pay attention that at university level, especially a defined set of pairs can be a definition not a representation of a function.

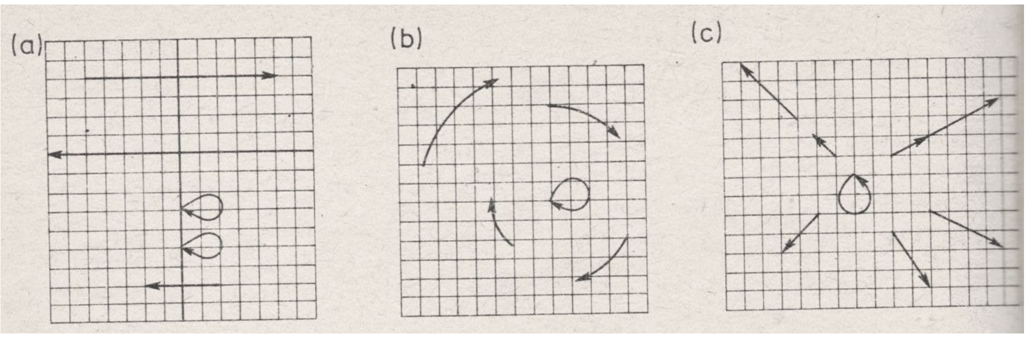
We add to the list of representations from the textbook the missing ones.

Lecturer provides the Teaser Video "Nomograms".

- Nomogram: https://www.funthink.eu/learning-environments/lower-secondary-education/nomogram-intro-and-graph

Activity 7. Let us list the representations of the notion of function

Pre-service teachers reflect individually on Handout 3.



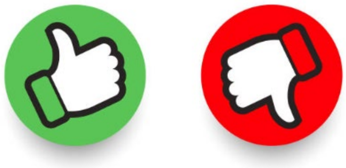
Source: Turnau, S. (1990). Wyklady o nauczaniu matematyki [Lectures on the teaching of mathematics], WSiP, Warszawa, p. 178

- Which transformation is represented by the following graphs?
- How do we name such graphs?

Summary:
- Are the transformations presented in this way functions?
  - Yes
- How do we name such graphs?
  - arrow-point graphs

Activity 8. Is the definition of a function correct?
Discussion on pre-service teachers own definitions (from Activity 4)

True-false method. The lecturer reads the anonymous definitions formulated in activity 4. Everyone shows their hand in front of them assessing the correctness of this definition. The lecturer indicates individuals to provide arguments for/against the correctness of the definition.



Activity 9. The lecturer can raise the topic of the linear function definition

Is the definition of a function correct?

Lesson Plan Teacher Course

Functional thinking: Aspects, tasks and representations	
Teaching hours:	180 minutes
Target group:	Pre-service teachers
Description	
Goals:	• Introduction to functional thinking • Aspects of functional thinking • Representations and tasks to support aspects of functional thinking
Structure:	• 20 minutes Talk on theoretical background of Formative assessment including two brainstorming • 20 minutes Group work – students' solutions analysis • 20 minutes Collective discussion about the students' solutions. • 30 minutes Standard misconceptions, errors and learning difficulties

Activities

Functional THINKING introduction (10 minutes)

Activity 1. Brainstorming
Think of the one word, which is the answer on the following question: What is the goal of teaching and learning mathematics.

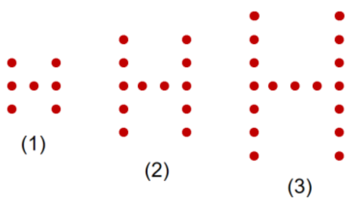
- The lector lets pre-service teachers think about the question and the answer for a little while. After this period, he collects the ideas from the group. Usually, “thinking” is one of the most frequent answers.
- A group discussion follows: What does it mean – that thinking is the goal of education? In next few sessions we will aim for deep understanding, how can we develop thinking – especially **functional thinking**.

Four tasks (45 minutes)

Activity 2. Solve the tasks in as many ways as possible

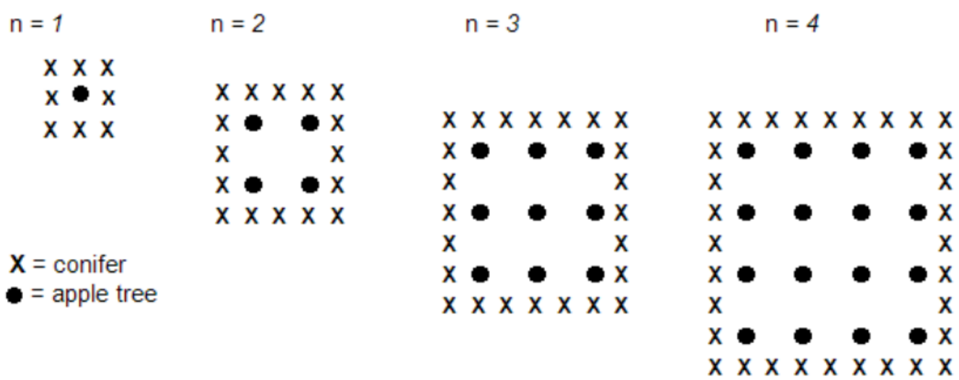
A. We have the following situation: “The candle is initially 24 cm tall. It shrinks by 2 cm every hour.” What formula would you use to express this situation? (adapted, original task by R. Nitsch, www.codi-test.de)

B. Emil has set the displayed first three patterns with circle cards. What will be the number of cards in the fifth pattern? What formula can be used in order to determine the number of cards for any of the patterns?



Adapted from a task of the Realschule final examination, Germany

C. A farmer plants apple trees in a square pattern. In order to protect the trees against the wind he plants conifers all around the orchard (see picture).



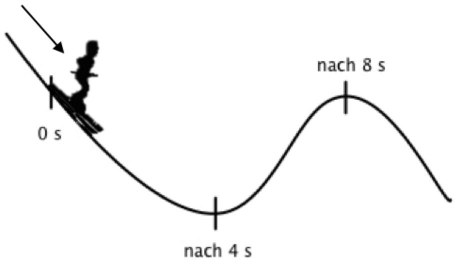
OECD, 2009, p. 102

Complete the table:

n	Number of apple trees	Number of conifers
1	**1**	**8**
2	**4**	**...**
...	**...**	**...**

When (for what *n*) will be the number of conifers same as the number of apple trees?

D. A skier already has an initial speed at the time 0 seconds and then let himself glide down the hill (without intentionally braking). When is he faster: at second 4 or 8? Describe in words how his speed changes with time. A second skier is traveling the same distance and has a higher initial speed than the first skier has. Compare the two runs in words.



cf. Barzel & Ruchniewicz, 2020, p. 9

- Candle and Skier tasks are to be solved by each of the groups; the other two can be split among the groups evenly.
- Each group uploads the pictures of their solutions into some kind of online shared visual space (e.g. OneNote, JamBoard, Padlet, …). It is a good idea for the upcoming discussion to ask groups to label each of the pictures somehow. For instance, Name of task – Number of group – Number of solutions (e.g. Candle_1_1).
- Based on our experience, it is also important to underscore that real solutions should be written. Some of the groups wrote just “using table”.

Activity 3. Solution analysis

Analyze the solutions provided by all groups. Answer the following questions:

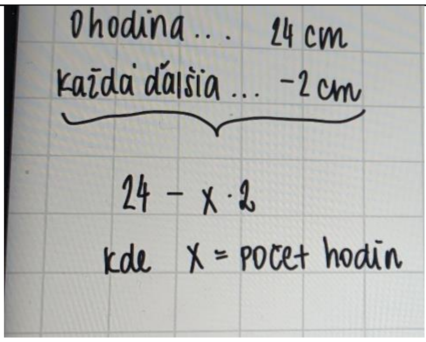
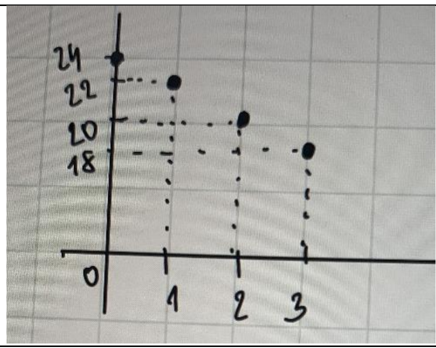
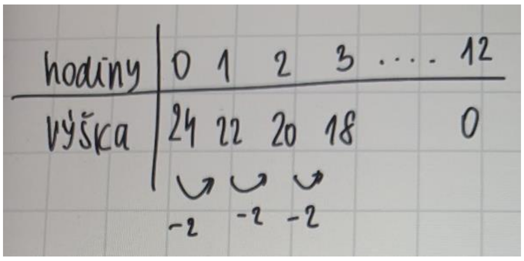
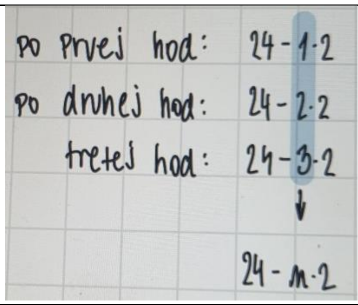
A. What are the differences between different solutions of the same tasks?

B. What makes solutions of different tasks similar?

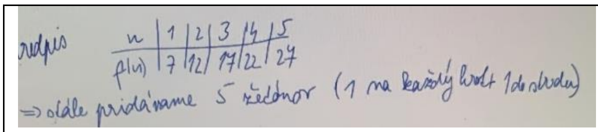
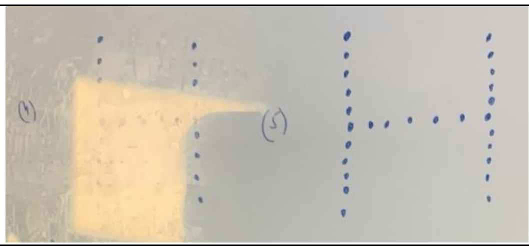
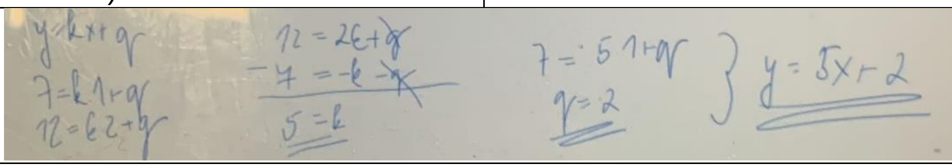
C. What knowledge and skills are needed to solve these tasks?

During the pilot phase, we collected the following solutions:

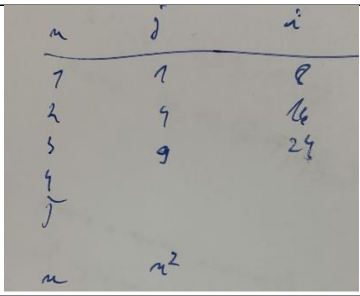
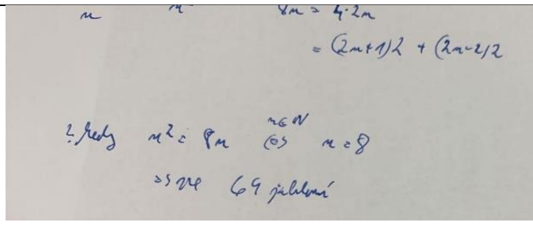
The Candle task:

<table>
<tr><td>
</td><td>
</td></tr>
<tr><td>0 hours … 24 cm
Each next … -2 cm
$24 - x * 2$?, where x = number of hours</td><td></td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>Hours
Height</td><td>After 1st hour:
After 2nd hour:
After 3rd hour:</td></tr>
</table>

The cards task:

<table>
<tr><td></td><td></td></tr>
<tr><td>We keep adding 5 buttons (1 for each end + 1 into the middle)</td><td></td></tr>
<tr><td colspan="2">
</td></tr>
</table>

The Orchard task

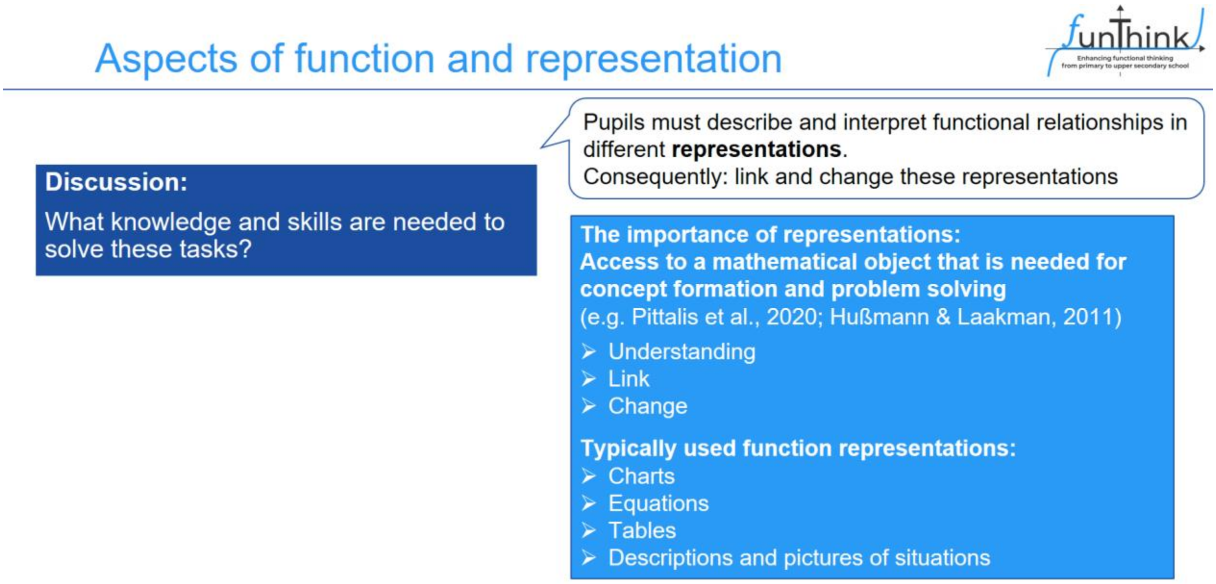
<table>
<tr><td></td><td></td></tr>
</table>

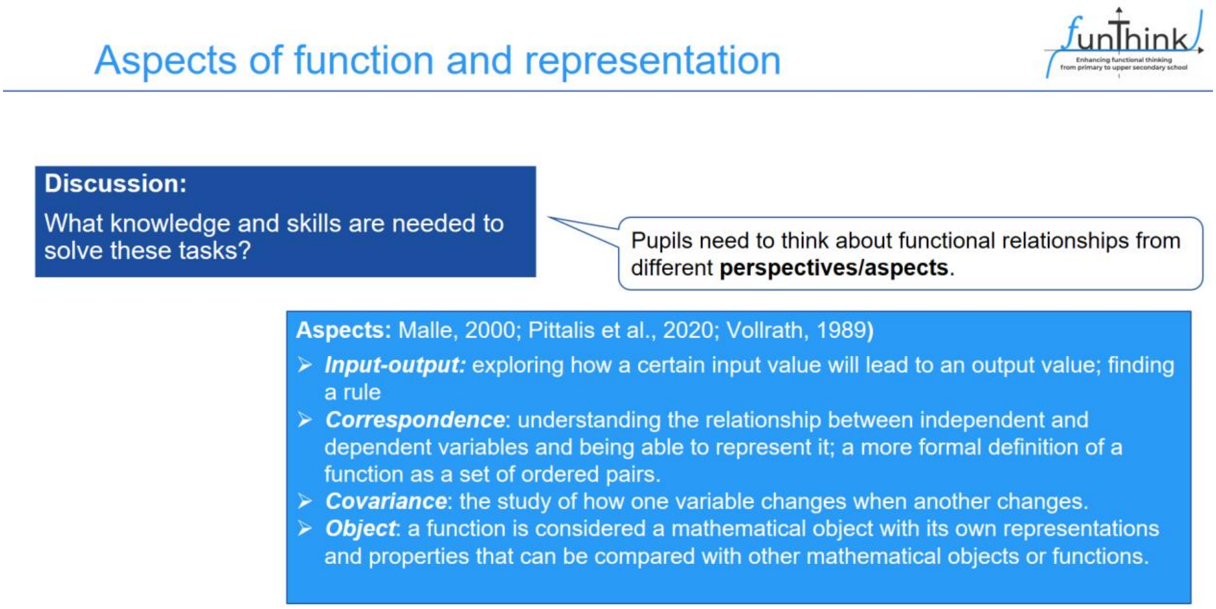
j – jablone – apple trees j – ihličnany – coniferas

The Skier task

Pre-service teacher will easily find that the differences (commonalities) are in the way of representing the tasks, the functions. In addition, some of our students said that there are different “ways of thinking”.

This “discovery” can help us to introduce representations and aspects:





Four aspects and representations (45 minutes)

Here, we suggest the lecturer takes over and explains in detail aspects, however, keeps the discussion with students vivid. He builds on the pre-service teachers' solutions of the Candle task.

Aspects of function and representation: input - output

Aspects: Malle, 2000; Pittalis et al., 2020; Vollrath, 1989)

- ***Input-output:*** exploring how a certain input value will lead to an output value; finding a rule.

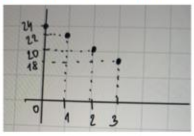
We have the following situation:

"The candle is initially 24 cm high. Every hour it shrinks by 2 cm."

Describe it mathematically.

Representations:

- Chain of numbers 24 -2 22 -2 20 -2 18 -2 ... ...
- Graphical: Point in the coordinate system



- Table without perceptions of any other relationships
- ~~General formula~~

Aspects of function and representation: correspondence

Aspects: Malle, 2000; Pittalis et al., 2020; Vollrath, 1989)

- ***Correspondence***: understanding the relationship between independent and dependent variables and being able to represent it; a more formal definition of a function as a set of ordered pairs.

We have the following situation:

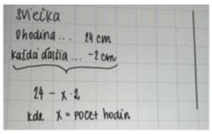
"The candle is initially 24 cm high. Every hour it shrinks by 2 cm." What equation would you use to express this situation?

Representations:

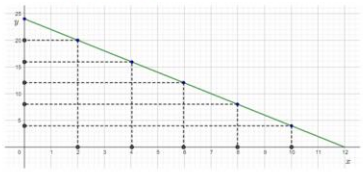
- Table - if the pupil perceives individual ordered pairs

Time	0	1	2
Height of the candle	24	22	20

- General regulation



- Graphic
  Graph a function where the pupil perceives ordered pairs of and

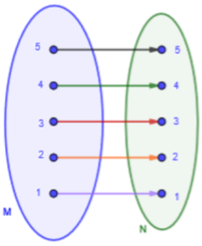


Aspects of function and representation: correspondence

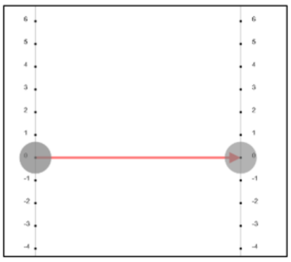
Aspects: Malle, 2000; Pittalis et al., 2020; Vollrath, 1989)
- ***Correspondence***: understanding the relationship between independent and dependent variables and being able to represent it; a more formal definition of a function as a set of ordered pairs.

We have the following situation:
"The candle is initially 24 cm high. Every hour it shrinks by 2 cm." What equation would you use to express this situation?

Representations:
- Venn diagram with arrows



- Nomogram

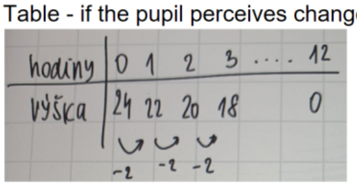


Aspects of function and representation: covariance

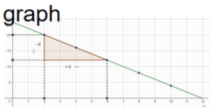
Aspects: Malle, 2000; Pittalis et al., 2020; Vollrath, 1989)
- ***Covariance***: the study of how one variable changes when another changes.

We have the following situation:
"The candle is initially 24 cm high. Every hour it shrinks by 2 cm." What equation would you use to express this situation?

Representations:
- Table - if the pupil perceives changes in individual variables



- Graphically - reading the change and rate of change from the graph



- Slope

Aspects of function and representation: object

Aspects: Malle, 2000; Pittalis et al., 2020; Vollrath, 1989)
- ***Object***: a function is considered a mathematical object with its own representations and properties that can be compared with other mathematical objects or functions.

We have the following situation:
"The candle is initially 24 cm high. Every hour it shrinks by 2 cm." What equation would you use to express this situation?

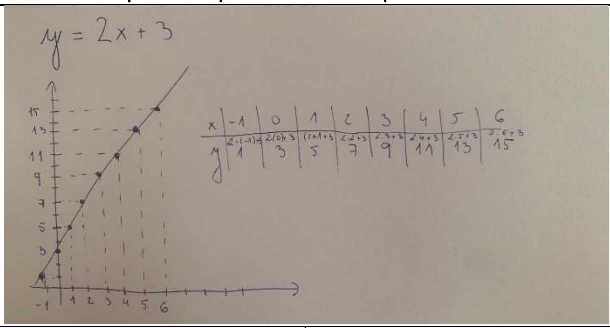
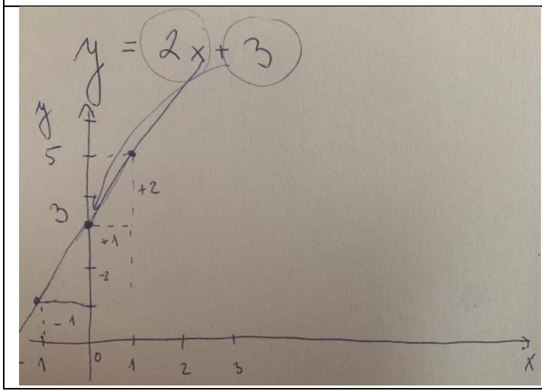
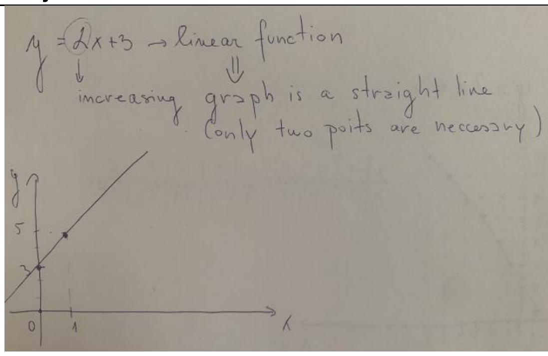
Representations:
- Formula - use of general formulas of functions (e.g. $y = ax + b$ is linear function)
  - Identification of the properties of a given function (decreasing - negative „*k*")
- Graphically - using the properties of a given class of functions (the graph of a linear function is a straight line)
  - Identifying the properties of a given function (decreasing - correct slope of the line)

Differences in tasks (45 minutes)

Activity 4. Use different aspects when solving the task in many different ways
Analyse the solutions provided by all groups. Answer the following questions:

Draw the graph of the following function: $y = 2x + 3, x \in \mathbb{R}$.

Possible solutions:

<table>
<tr><td colspan="2">Input-output & Correspondence</td></tr>
<tr><td colspan="2">
</td></tr>
<tr><td>Covariation</td><td>Object</td></tr>
<tr><td>
</td><td>
</td></tr>
</table>

Activity 5. Your solutions analyses
Analyze the solutions of Cards task, Orchard task and Skier task. Identify, which aspects were used in each of the solutions.

- You can adjust the presentation before this meeting – enrich it with PSTs solutions
- A group discussion:
  - What is the difference between the Skier task and all other tasks? —> Some tasks are easily solved by using different aspects, some tasks require dominantly one of the aspects.
  - In the situation of choosing the task for a lesson, which are better? —> It depends on the goal of the task.

Questions for reflection:

- Which tasks would you choose if you wanted **to see which aspects do students naturaly prefer?**
  A. Candle
  B. Buttons
  C. Orchard
  D. Skier

- Which tasks would you choose if you wanted **to find out whether pupils can use covariance?**
  A. Candle
  B. Buttons
  C. Orchard
  D. Skier

Activity 6. Back to the "set of tasks"

A. Which tasks would you choose if you wanted to see input-output aspect in the solution?

B. Which tasks would you choose if you wanted to see covariational aspect in the solution?

C. Which tasks would you choose if you wanted to see correspondence aspect in the solution?

D. Which tasks would you choose if you wanted to see object aspect in the solution?

E. Which tasks would you choose if you wanted to see which aspects do students prefer?

- At this activity we use set of tasks from the Knowledge of Topic module
- Pre-service teachers work on this task independently, afterwards they compare their answers in small groups.
- Whole-group discussion:
  - A:
  - B: 3, 4, 5, 9
  - C: 12b
  - D: 6, 8, 15
  - E: 1, 2, 10, 14

Summary (45 minutes)

Activity 7.
In a group, create a digital poster about one of the aspects of the functional thinking. Be sure to include important information: representations, tasks, solutions, etc.

- Pre-service teachers form at least four groups, the lector assigns them the aspects randomly.
- They have 30 minutes to create their posters. They can be encouraged to use particular software or to make their own choice (PowerPoint, Canva, MS Sway, GoogleSlides, …)
- During the last 15 minutes, the groups present their posters.
- The lector acknowledges good presentations and corrects incorrect concepts.

Lesson Plan Teacher Course

Curriculum	
Teaching hours:	2 x 90 minutes
Target group:	Pre-service teachers
Description	
Goals:	• Participants gain insights into functional thinking in the curriculum • Participants know how functional thinking can develop • Participants know how functional thinking plays a role in other subjects
Structure:	• 10 minutes Brainstorming on “what does it mean to develop functional thinking” • 80 minutes Group work • 90 minutes Collective discussion about development of functional thinking in terms of aspects, representations and applications.

Activities

Brainstorming (10 minutes)

Activity 1. What does “Development of functional thinking” mean?

Write on the board everything what comes in your mind when you think of development of functional thinking.

Slides 1-3 in the presentation.

The lector writes **Development of functional thinking** on the board and asks pre-service teachers to write down everything what is in their minds connected to the heading. This activity serves on one hand as a starter of the following discussion, however it is also a tool to revise what was said about functional thinking so far.

Possibly, the lector can use some online solution of collecting ideas from the individual pre-service teachers. In this way, he will have overview of individual pre-service teachers understanding of functional thinking.

The lector creates a summary of the ideas written on the board (or collected via online tool) and – if possible – highlights the following three categories:

ASPECTS

- perceive the concept of function in its diversity
  - input-output
  - covariance
  - correspondence
  - object
- using the appropriate aspect for the given situation

REPRESENTATIONS

- understanding different forms of representations
  - graph
  - formula
  - table
  - verbal description
  - ...
- change of representation
- choosing the appropriate representation considering the situation

APPLICATIONS

- see and use functions and functional thinking within mathematics
- see and use functions and functional thinking outside of mathematics

Group work (80 minutes)

Activity 2. Curricular material study
Each group is assigned a type of school which is used for the following tasks.

A. Study the curriculum documents for the given type of school.

B. Distribute the 12 points between the aspects of function according to their importance at the given type of school / grade level.

C. For each representation, indicate whether it is used at the given type of school/ grade level
(**P** – used, **I** – used "introductory", "preparatory", "propaedeutic", **X** – not used).
In the free space, explain important information. For example, what kind of propaedeutic is used, why the representation is not used, etc. Moreover, state what other representations of functions occur.

D. List specific mathematical topics in which functional thinking can be developed. Explain how.

- within the scope of the function
- out of scope of function

E. List several practical applications of functions appropriate for the given type of school/ grade level. Focus also on the curriculum of other school subjects.

To make the following discussion efficient, we share online the excel document developed for this activity. Each group will fill one of the sheets. Their ideas will be collected in the overview sheet automatically. This technological solution is not necessary, however, very helpful.

Our experience says that pre-service teachers need to be encouraged to be specific enough concerning applications (tasks E, D). We can motivate them to elaborate these tasks deeply as this material can serve them in the future when thinking about lesson plans concerning functions.

Task E can be restricted to some of the subjects, if lector assume it will be more beneficial.

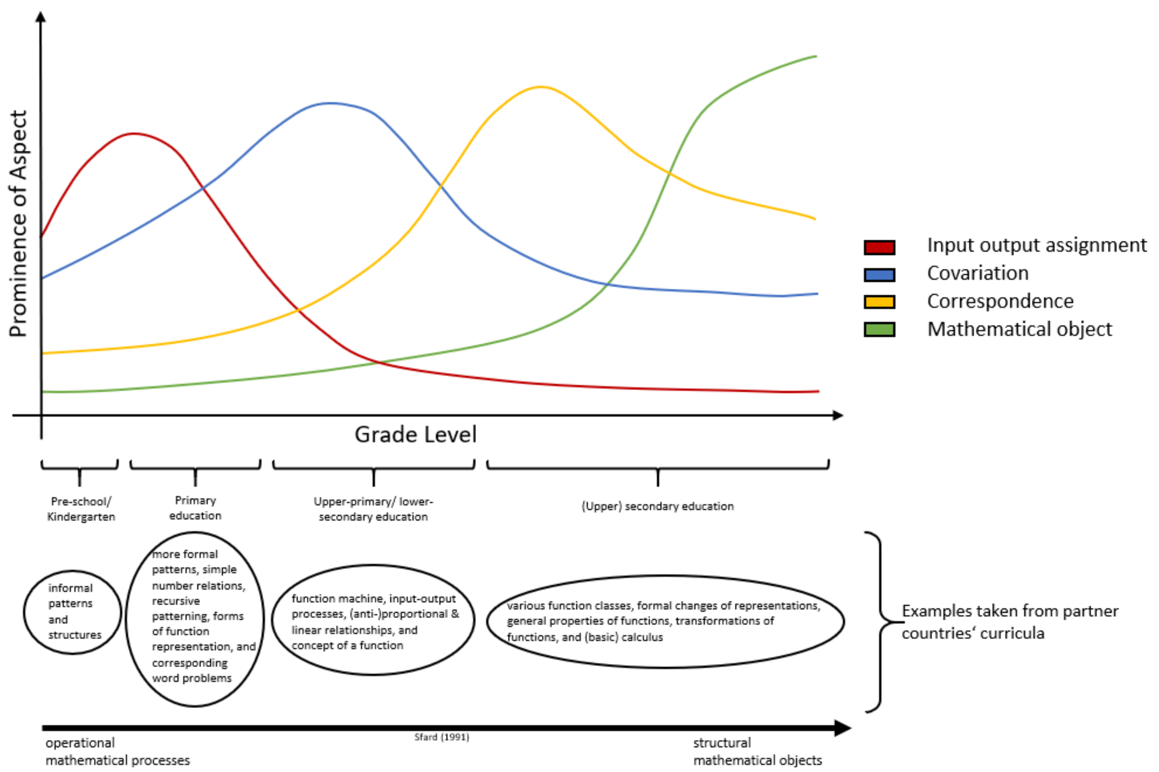
In the next part of the session, we will discuss what pre-service teachers revealed in the curricular documents. The discussion is, thus, dependent on their activity and on the curriculum of the country. The following paragraphs serve as guideline for the lector to structure the discussion reasonably. Moreover, we provide some activities, which might help in the case the discussion is over too fast or some ice-breaker is needed.

Collective discussion: Aspects (30 minutes)

Slides 5-8 in the presentation. Firstly, it is important to discuss pre-service teachers ideas (add the information from the overview sheet) and subsequently discuss the perspective of the lecturer.

The table needs to be adjusted depending on the national school system prior to discussion.

	primary	low-secondary	high-secondary (continue with job)	high-secondary (continue with university without mathematics)	high-secondary (continue with university with mathematics)
Input-output					
Covariation					
Correspondence					
Object					



Activity 3. Aspects

Explain four aspects of the function on the following functions:

A. $y = \sin x \, , \, x \in \mathbb{R}$

B. $y = \cos x \, , \, x \in \mathbb{R}$

Collective discussion: Representations (30 minutes)

Slides 9-11 in the presentation.

Firstly, we see what pre-service teachers found in the curricular documents and textbooks. Fill the empty table in the presentation or display the online document. The following table depicts the situation is Slovakia and need to be adjusted to describe the situation in the country of use.

	primary	low-secondary	high-secondary (continue with job)	high-secondary (continue with university without mathematics)	high-secondary (continue with university with mathematics)
Graph	I	P	P	P	P
Formula	I	I - P	P	P	P
Table	P	P	P	P	P
Word description	I	I - P	P	P	P
Other	chain, nomogram, venn diagrams with arrows				

Activity 4. Representations

Represent the function $y = 5x + 10,\ x \in \mathbb{R}$ in as many ways as possible. Explain, how different representations pinpoint different aspects. Discuss, what are the benefits and restrictions of these representations.

Collective discussion: Propedeutics to representations (5 minutes)

Graph - propedeutics

- **What is needed to be able to use the graphical representation correctly?**
- **Developing the concept of number**
  - $\mathbb{N} \longrightarrow \mathbb{Q}^{+} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{R} \longrightarrow \mathbb{C}$
  - When is the number line "full"?
- **Coordinate system**
  - propedeutics - coordinates in the chessboard and filling in the table
  - problems with the order:
    - **in the chessboard A1 and 1A are the same square**
    - **in a table, many people prefer the row (y-coordinate) over the column (x-coordinate)**
  - a square in the chessboard/table vs. a point in the coordinate system

Formula - Propedeutics

What does it take to know how to use a formula correctly?

- **Different understandings of the letter in mathematics**

1. **The letter represents one specific** value
denotes an unknown number, which we determine by logical reasoning or guessing (not using equivalent equation modifications), for example What number should we substitute after x to make the equation hold: x-5=12.
2. **We don't need to** know **the value of the letter**
For example, in the problem: For the numbers x and y, x+y=10. What is x+y-8=?
3. **Letter as a designation**
is an agreed designation, a naming convention, e.g., the letters a, b, c denote the sides of a triangle, the letter t denotes a tone, and so on. For example, 3f=1y, where f denotes feet and y denotes yards.
4. **Letter as one unknown** value
a letter represents one specific number, we often talk about an unknown that can occur in equations or when modifying expressions, for example 3x-5=4x. When working with the letter in this context, for example, we use a variety of methods for modifying equations, including equivalent modifications.
5. **The letter takes more than one value (any value from the definition field**)
a letter here denotes several numbers (even infinitely many), as in the inequality 4x-5<15. Meanwhile, when working with a letter at the level of a letter as an unknown, for example when modifying equations, we often use a letter in this context.
6. **Letter as a constant**
e.g., π, e.
7. **Letter as parameter**
for example, the letters k and q in the notation of the general prescription of the linear function y=kx+q
8. **Letter as a variable**
In this context, pupils encounter the letter in the context of the concept of function, which also appears implicitly in school mathematics before its introduction, for example, in the context of exploring the dependence of path on time.

Collective discussion: Thematic area of functions in the curriculum (10 minutes)

Thematic area of functions in the curriculum

primary	low-secondary	high-secondary (continue with job)	high-secondary (continue with university without mathematics)	high-secondary (continue with university with mathematics)

Thematic area of functions in the curriculum

primary	low-secondary	high-secondary (continue with job)	high-secondary (continue with university without mathematics)	high-secondary (continue with university with mathematics)
Propedeutics	Direct and indirect proportionality Linear function			

Thematic area of functions in the curriculum

- **Which properties of a function or concepts can be discussed for which functions?**

Linear function	Monotony Inverse function
Power functions	Evenness / Oddness
Goniometric functions	Periodicity
Exponential, Logarithmic	Monotony Inverse function

Functions in other areas of the mathematics curriculum

primary	low-secondary	high-secondary (continue with job)	high-secondary (continue with university without mathematics)	high-secondary (continue with university with mathematics)

Functions in other curriculum areas

- **Geometric formulas (perimeters, contents, ...) as functions**
  - Creating tables
  - Noticing differences and similarities between tables (the content of a square grows faster than its content)
- **Conformity and similarity of formations**
  - Ratio and direct proportionality
- **Analytical equations of some geometric figures**
  - Quadratic function - parabola, Linear function - straight line, ...
- **Equations and inequalities**
  - Graphical solutions of equations and inequalities
  - Using properties of functions in reasoning about modifications (change of sign because the function is decreasing; non-equivalence of the modification because the function is not simple, ...)
- **Statistical models**
  - Linear function as a regression line

Collective discussion: Applications (30 minutes)

The lector needs to study curricular documents of other subjects in before the meeting to add what was not mentioned by students.

Activity 5. Where can you see the functions

You have 10 minutes to walk across the university building and write down the list of all observed functional dependencies. The group with the longest and at the same time meaningful list wins!

Functions in other subjects - examples of applications

primary	low-secondary	high-secondary (continue with job)	high-secondary (continue with university without mathematics)	high-secondary (continue with university with mathematics)

Functions in other subjects - low-secondary

- **Chemistry:**
  - calculate the mass fraction of a component in solution; the mass of solute, solvent and solution
  - carry out experiments to measure thermal changes in chemical reactions, record the results of the experiments in tables and interpret them,
- **Geography:**
  - describe the apparent path of the Sun and Moon in the sky (pictures, sketches),
  - determine from the map of time zones where on Earth there are more hours than in Slovakia and where there are fewer,
  - identify a selected location on a map using geographical coordinates
  - compare distances on maps of different graphical scales,
  - Talk about balloon travel from the equator to the polar countries -> summarize changes in the air with increasing altitude

Functions in other subjects - low-secondary

- **Physics:**
  - analyse graphs, explain the graph progression, process the measured values in the form of a graph (in general)
  - construct a graph of the linear dependence of the trajectory on time for a uniform rectilinear motion
  - construct a graph of the constant velocity versus time for uniform straight-line motion
  - construct a graph of the direct proportionality between current and voltage from the measured values
  - linear dependence of the gravitational force and the mass of the body
  - graphical representation of velocity and trajectory over time
  - constructing a graph of electric current versus electric voltage
- **Informatics:**
  - information in tables, cell, relationships between cells, graphs

Functions in other subjects - high-secondary

- **Chemistry:**
  - solve problems to calculate the mass fraction, and the concentration of a component,
  - sort a group of elements into elements with small and large electronegativity values based on their location in the PTP
  - Calculate the mass of the reactant or product based on the chemical equation, given the mass of the solid product or reactant,
  - determine the value of the heat of reaction of the reverse reaction based on the value of the heat of reaction of the direct reaction using the 1st thermochemical law
  - compare the rate of chemical reactions by observation
  - Explain the nature of the effect of changes in temperature, reactant and catalyst concentration on the rate of a chemical reaction,
  - explain the nature of the effect on the equilibrium state of a system of adding a reactant or removing a product, changing temperature and pressure

Functions in other subjects - high-secondary

- **Geography:**
  - determine the location of any place on the map using geographical coordinates
  - retrieve and interpret statistics, facts and relevant facts from reliable information sources
  - creatively use geographical knowledge in various graphic forms (content of thematic maps, tables, diagrams, charts, cartograms, carto-diagrams)
  - understand and use adequately the data presented in GPS instruments and navigators
  - calculate the actual distance of places on a map from a numerical or graphical scale
  - correctly interpret climate data presented in various graphical and textual forms (tables, charts, graphs, climate diagrams, thematic maps),
  - know the meaning and reliability of meteorological forecasts,
  - compare the influence of internal and external geological processes on the formation of the Earth's surface
  - correctly interpret information on population development and composition presented in the form of graphs, tables, age pyramids, diagrams and thematic maps,
  - correctly interpret statistical data and economic indicators of the economic performance of countries and regions of the world
  - correctly interpret statistical data and economic indicators of the economic performance of individual regions, states
  - correctly interpret information on the development and composition of the region's population presented in the form of graphs, tables, age pyramids, diagrams and thematic maps
  - correctly interpret information on the development and composition of the population of Slovakia, presented in the form of graphs, tables, age pyramids, diagrams and thematic maps
  - estimate the development and possible risks of changes in key industries in Slovakia,

Functions in other subjects - high-secondary

- **Physics:**
  - analysis of the graphical representation of the dependence of the trajectory on time (uniform, non-uniform, decelerated and accelerated movements)
  - interpret the slope of a graph of linear dependence and the intersections of the graph with the coordinate axes
  - display of the work done in a force versus displacement graph
  - free fall - constructing a graph of path versus time, determining the dependence of speed on time
  - temperature dependence of electron motion, temperature dependence of electrical resistance
  - linear dependence, linear dependence graph
  - temperature dependence of metallic conductor resistance
  - harmonic oscillatory motion (the time diagram of the oscillatory motion is a sinusoid or cosine-sinusoid)

Functions in other subjects - high-secondary

- **Informatics:**
  - analyze the problem, propose an algorithm for solving the problem, write the algorithm in an understandable formal form, verify the correctness of the algorithm
  - understand the finished programs, determine the properties of inputs, outputs and relationships between
  - them
  - solve tasks using commands with various constraints on the use of commands,
  - variables, types and operations
  - input and output of information depending on its type
  - spreadsheet calculator - formula, function
  - input and output devices
  - problem, algorithm, algorithms from everyday life, ways of writing an algorithm
  - programming language - commands (assignment, input, output), variables

Lesson Plan Teacher Course

Learning Environments: Introduction	
Teaching hours:	90 minutes
Target group:	Pre-service teachers
Important links	
Videos	Teaser video for chosen learning environment
Description	
Goals:	• Learn about the four design principles (short overview, more later). • Know the structure of the learning environments with handouts and teacher guide. • Identify the learning objectives of the learning environments and link them to the aspects of functional thinking.
Structure:	• 15 minutes Talk on theoretical background of design principles and structure of learning environments • 40 minutes Partner work – introduction to a chosen learning environment • 20 minutes Collective discussion about the learning environment • 15 minutes small group activity – development of further tasks

Activities

Learning environment introduction (15 minutes)

Activity 1. Theoretical Background

- The lector provides a first overview of the design principles
- The lector introduces the structure of the learning environments

Introduction to first learning environment (40 minutes)

The lecturer choses a learning environment to explore in the next part of the course. Do not use the learning environments nomogram, walking graphs, number line or distance-time as they will be included in the course later. The printed teacher guide will serve as a handout but the first page of the teacher guide is not handed out at the beginning, it can be done after the discussion.

Activity 2. Exploration

Getting to know the learning environment chosen by the lecturer.

- Familiarise yourself with the teacher guide.
- Work on the tasks in the handout of the learning environment.
- Answer the following questions:
    - What prior knowledge do you expect for the learning environment?
    - What are the learning objectives of this learning environment / the different tasks?
    - Which aspects of functional thinking are addressed in this module?
    - What forms and changes of representation are focused on in this learning environment?

- The lector chooses a learning environment for this exploration and makes it available for the pre-service teachers. ().
- After the exploration, the questions are discussed in the whole group. The fist page of the teacher guide can serve as basis for the discussion for the lecturer.
- If student answers are available for the chosen module, they can be handed out and discussed afterwards.

Activity 3. Further tasks

Create further tasks to match the learning environment chosen by the lecturer.

Provide details of:

- Learning objective(s)
- Addressed aspects of functional thinking
- Representations used or changes in representation

This activity can be implemented if time is available. Students work in small groups and develop tasks for the discussed learning environment. These tasks can be both practice tasks and more advanced tasks.

Lesson Plan Teacher Course

Situatedness as design principle	
Teaching hours:	90 minutes
Target group:	Pre- and in-service teachers
Important links	
Classroom videos:	Teaser Video Patterns
Description	
Brief Description:	Teachers compare, analyze and design activities in respect to the notion of inquiry-based learning in mathematics
Learning Objectives:	• Identify, discuss and reflect the characteristics of teaching that builds on inquiry-based learning • Design activities that address inquiry-based learning as a design-principle in functional thinking situations (such as investigating the aspect of slope in linear situations)
Pedagogical Content Knowledge:	• How can inquiry-based learning be integrated in functional thinking activities? • How can key aspects of functional thinking (such as slope) be developed based on inquiry-based learning?
Structure:	• Introduction task → teachers compare different textbook tasks • Whole class discussion and brainstorming: to what extent does each of the three textbook approaches engage students to an authentic inquiry-based learning experience? • Presentation on inquiry-based learning • Watching teaching episode • Group work

Activities

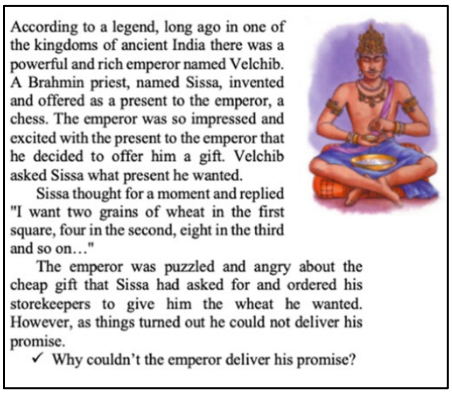
Introduction task. (30 min)

In the following section we present the introduction of the concept of powers-exponents in three different teaching textbooks.

i. Compare the didactical approaches of the three textbooks.
ii. Which one(s) would you use? Which one(s) can build on students' prior knowledge and experience. Explain.

<u>Introduction of Powers and Exponents:</u>

Textbook A

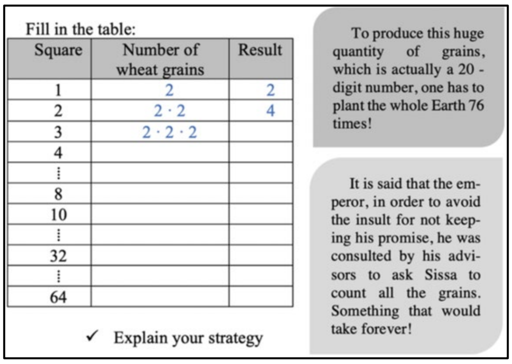


According to a legend, long ago in one of the kingdoms of ancient India there was a powerful and rich emperor named Velchib. A Brahmin priest, named Sissa, invented and offered as a present to the emperor, a chess. The emperor was so impressed and excited with the present to the emperor that he decided to offer him a gift. Velchib asked Sissa what present he wanted.

Sissa thought for a moment and replied "I want two grains of wheat in the first square, four in the second, eight in the third and so on…"

The emperor was puzzled and angry about the cheap gift that Sissa had asked for and ordered his storekeepers to give him the wheat he wanted. However, as things turned out he could not deliver his promise.

✓ Why couldn't the emperor deliver his promise?



Fill in the table:

Square	Number of wheat grains	Result
1	2	2
2	$2 \cdot 2$	4
3	$2 \cdot 2 \cdot 2$	
4		
⋮		
8		
10		
⋮		
32		
⋮		
64		

✓ Explain your strategy

To produce this huge quantity of grains, which is actually a 20 - digit number, one has to plant the whole Earth 76 times!

It is said that the emperor, in order to avoid the insult for not keeping his promise, he was consulted by his advisors to ask Sissa to count all the grains. Something that would take forever!

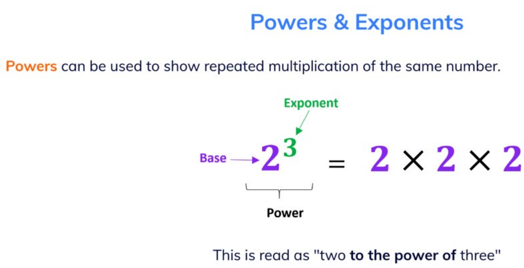
Textbook B

Use your calculator to complete the following table

	Result		*Result*
$2 \cdot 2$		2^2	
$2 \cdot 2 \cdot 2$		2^3	
$2 \cdot 2 \cdot 2 \cdot 2$		2^4	
$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$		2^5	
$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$		2^6	

(a) What do you observe?
(b) How can we express repeated multiplication of the same number? Provide examples.

Textbook C



Textbooks: Athanasiou et al., 2016a, p. 47f & 2016b, p. 17

Whole class-discussion (15 min)

(Pre-service) teachers discuss to what extent each of the three textbook approaches could engage students to an authentic inquiry-based learning experience.

Brainstorming in whole class (15 min)

- What are the characteristics of an activity/lesson in the framework of inquiry-based learning?
- Teachers give examples from their experience.

Presentation (15 min)

The instructor presents a proposed teaching model that builds on inquiry-based learning (see presentation design principle inquiry-based learning)

Watching teaching episode – Discussion (30 min)

Teachers watch a five-minute teaching episode regarding the introduction of Patterns (CYPRUS VIDEO FOR PATTERNS -GRADE 6)

Discussion regarding the extent to which the adopted activity and the orchestration of the discussion promoted inquiry-based learning. Teachers suggest possible modifications.

Group work (15 min)

Teachers suggest an Exploration-Investigation that addresses the following:

> *Students in grade 8 are to discover that the slope of a line is equal to the change in the y-coordinate divided by the change in the x-coordinate*

Teachers suggest an appropriate scenario with the use of the following app:

https://www.geogebra.org/classic/mvvajgsr

This is followed by a presentation of the developed approaches developed by particular group of teachers.

Lesson Plan Teacher Course

Design Principles: Embodiment	
Teaching hours:	90 minutes
Target group:	Pre-service teachers
Description	
Goals:	• Pre-service teachers experience embodied design tasks • Pre-service teachers learn about underlying ideas of embodied design • Pre-service teachers learn about nomograms • Pre-service teachers experience embodied design and other tasks on nomograms • Pre-service teachers reflect on how to implement embodied design task in classroom
Structure:	• Hands-on experience with embodied design • Input on nomograms and embodied design • Hands-on experience with embodied design nomogram tasks • Reflecting on the experience with the learning environment, with a focus on embodied design principles

Activity 1.

Embodied design: introduction

Estimated duration: 20 minutes

Materials: each pair of participants needs a touch-screen device (phone, tablet, laptop).

Participants explore the tasks at the website: https://embodieddesign.sites.uu.nl/activity/, taking care to follow the instruction.

- Move the points (or one point) and make the feedback green.
- Find another green position.
- Keep the feedback green (move the points in a way that the feedback would be green all the time). It is rather challenging but possible in each case!
- When you are fluent (let your "body" practice at first!), reflect what is the rule that determines the green colour of the feedback.
- Move to the next task by listing the tasks on the bottom of the screen.

Participants make notes on three levels:

- what they experience themselves when they perform the task
- a didactical perspective: how the task might relate to mathematical learning
- what are the common characteristics of this design genre

Activity 2.

Input on embodied design

Estimated duration: 15 minutes

The instructor gives an interactive lecture on embodied design. The instructor addresses at least the following aspect of embodied design

- What is embodied cognition? The theory of embodied cognition states that cognition cannot be separated from the perception and motor systems in the body. Or positively phrased, cognition is fundamentally intertwined with and dependent on action and perception.
- Why embodied cognition in mathematics education? Learning how to bike is an informative example of embodied cognition. Most of the learning is done by so-called loops between perception and action, not necessarily accompanied by much reflection. Learning how to move in new ways like that is done in a way that seems very different from mathematics. However, embodied design explores precisely how this form of learning can be used in the mathematics classroom.
- What is embodied design for mathematics education? Embodied design tasks consists of three phases:

- (1) a motor problem where students learn to move in a new way.
- (2) A reflection on this movement: what is the rule (that keeps the feedback green)?
- (3) A mathematization of the rule: how can the rule be described by mathematical means?

Shapiro, L., & Stolz, S. A. (2019). Embodied cognition and its significance for education. Theory and Research in Education, 17(1), 19–39. https://doi.org/10.1177/1477878518822149

Abrahamson, D. (2009). Embodied design: constructing means for constructing meaning. Educ. Stud. Math. 70, 27–47. https://doi.org/10.1007/s10649-008-9137-1

Alberto, R., Shvarts, A., Drijvers, P., & Bakker, A. (Accepted/In press). Action-based embodied design for mathematics learning: A decade of variations on a theme. International Journal of Child-Computer Interaction, [100419]. https://doi.org/10.1016/j.ijcci.2021.100419

Participants use their notes to reflect on the theory.

Activity 3.

Hands-on experience with embodied design nomogram tasks

Estimated duration: 40 minutes

Material: each pair of participants needs a touch-screen device (phone, tablet, laptop).

In pairs participants work on the following module introducing the notion of nomograms: Nomogram Intro (10 minutes).

After this the instructor explains how the light context provides a meaningful situation to support students' thinking and reasoning about nomograms (10 minutes). In particular,

- How it should be read from left to right, object to shadow, input to output
- How there are rays (i.e. arrows) everywhere, but only some are drawn
- How they represent a fixed rule (in advanced language: function)

The instructor addresses how nomograms could support functional thinking as new representation of relations/function

- Emphasizing the input-output character
- Allowing an image of the meaning of inverse functions
- Allowing an image of composition of functions
- Giving an additional image of what it means for the formula/equation to be proportional (parallel arrows), or linear and non-additive (all arrows go through one point), constant (all arrows go to one point on the output axis).
- Giving a new image for domain and range: interval where arrows depart respectively arrive

Next, the participants explore two modules: Embodied Design Tasks Nomogram and Embodied Design Tasks Nomogram-Graph relation (20 minutes).

Participants make notes on three levels on a large sheet:

- what they experience themselves when they perform the task
- a didactical perspective: how the task might relate to mathematical learning
- what are the common characteristics of this design genre

Each duo writes at least three comments. The sheets are hung on the wall after the 20 minutes.

Activity 4

Reflecting on the experience with the learning environment

Estimated duration: 15 minutes

The group compares the comments on the wall, beginning with the most commonly mentioned. The goal is to answer the following questions:

- Why has the designer of the tasks chosen for these designs?
- Would these tasks improve students functional thinking?
- What is the interplay between embodiment and functional thinking? Do the aspects of embodiment in these tasks foster/support the development of functional thinking?

In the end of the discussion three participants each try to answer these questions summarizing the discussion before.

Lesson Plan Teacher Course

Digital Tools	
Teaching hours:	90 minutes
Target group:	Pre-service teachers
Important links	
Videos	Teaser video walking graphs Implementation videos: - Zig-zac - Walking graphs exploration
Description	

Goals:	• Introduction with hands-on experience: learning environment walking graphs
	• Reflection on the experience with the learning environment
	• Input:
	  • Tool use as a design principle
	  • Experiments with real and digital tools to foster functional thinking
	• Reflection on the learning environment with a focus on tool use

Structure:	• 40 minutes hands-on experience: LE walking graphs • 15 minutes reflection LE walking graphs with implementation videos • 20 minutes input tool use and experiments with real and digital tools • 15 minutes reflection on learning environment walking graphs with focus on tool use

Activities

Hands-on experience and reflection:
Learning environment walking graphs (55 minutes)

Activity 1: With help of the provided materials (handout from learning environment), experience the learning environment walking graphs

The learning environment walking graphs is used for the introduction to the design principle of tool use. In the learning environment itself, students experience the situation physically first and the digital applets are used later in the learning environment. For the experimentation in the teacher course, pre-service teachers experience the walking graphs situation physically (sensor and web based version) first and later digitally.

Depending on the size of the group, the group is split into two or three subgroups (no more than 30 (pre-service) teachers in one group). All groups complete the tasks according to how they are displayed on the corresponding slide.

The exploration is followed by a discussion.

Activity 2: Reflecting the LE with help of implementation videos

For this activity, two learning videos are available (Zig-zac and walking graphs exploration). Corresponding questions and tasks for discussion are included in the slides.

Input on tool use and experiments (20 minutes)

The lecturer provides information on the use of (digital) tools to support the learning. In a first task, the (pre-service) teachers reflect on the dimensions used in the learning environment walking graphs.

In a second part, the lecturer highlights how real and digital experiments can support the learning of functional thinking.

Reflection with focus on tool use (15 minutes)

After the hands-on experience and the input on tool use, the final step includes a reflection and discussion of the learning environment with a special focus on tool use.

Lesson Plan Teacher Course

Situatedness (LE: number line and distance-time (turtle))	
Teaching hours:	90 minutes
Target group:	Pre-service teachers
Important links	
Videos	-
Description	
Goals:	• Introduction with hands-on experience: learning environments number line and distance-time (turtle) • Reflection on the experience with the learning environment • Input: Situatedness as a design principle
Structure:	• 20 minutes input situatedness with small group activity • 50 minutes reflection LE walking graphs with focus on tool use • 20 minutes Exploration of further LEs

Activities

Activity 1: Identification of horizontal and vertical mathematization processes

As indicated on the corresponding slide, the (pre-service) teachers are asked to identify horizontal and vertical mathematization processes in the provided task and student solution.

Examples for each mathematization process are indicated in the notes in the PowerPoint presentation.

Activity 2: Exploring and reflecting the LEs

After some theoretical insights into the design principle of situatedness, pre-service teachers get to explore further learning environments: number line and distance-time (turtle).

For this activity, a number line, tablets, the handouts and teacher guides (available on website) are needed. If possible, the group should be divided into two subgroups.

During the exploration, the following questions should be kept in mind for later discussion:

- What are the learning goals of the learning environments and which aspects of functional thinking are addressed?
- What prior knowledge is required?
- What learning difficulties do you anticipate?
- Do mathematization processes take place and if so, which ones?
- To what extent are the activities being worked on situated?

The exploration is followed by a discussion with the whole group.

Activity 3: Exploration of additional LEs and development of further activities

If time is available, the learning environments double number line and function machine can be explored and discussed by the pre-service teachers. In addition, they are asked to develop further activities for one of the learning environments. The goal of the activities should be further development of functional thinking using the design principles.

Lesson Plan Teacher Course

Curriculum	
Teaching hours:	90 minutes
Target group:	Pre-service teachers
Description	
Goals:	• Pre-service teachers are familiar with "formative assessment" term • They are able to analyze students' solutions from the perspective of formative assessment. • They are familiar with misconceptions and learning difficulties concerning functions.
Structure:	• 20 minutes: Theoretical background of formative assessment including two brainstorming activities • 20 minutes: Group work – analysis of student solutions 20 minutes: Collective discussion about the students' solutions. • 30 minutes: Standard misconceptions, errors and learning difficulties

Activities

Brainstorming and explanation (20 minutes)

Activity 1. What comes to your mind when you hear: "assessment"?
Write on the board everything that comes to your mind when we talk about assessment.

Slides 2-12 in the presentation.

The lector writes **Assessment** on a board and asks pre-service teachers to write down everything that comes to their minds connected to the topic. This activity serves as the starter for the next discussion. However, it is also used as a tool to understand pre-service teachers' prior understanding of the topic.

The lector summarizes? makes a review of the ideas written on the board. In the next explanations, he/she tries to underscore the connections between the topic presented and the ideas mentioned by pre-service teachers.

Reminder: Inquiry

- 5E – Engage, Explore, Explain, Elaborate, **Evaluate**
- Why is the evaluation inner part of inquiry-based education? Does the teacher always need to assign grades?

Assessment is communication

- The teacher **gives informatio**n to the student what is the goal of teaching mathematics.
- The student **informs** how he fulfils these goals.
  - To whom is this information directly accessible?
    - For the given pupil: an opportunity for self-assessment
    - To another student: an opportunity for peer assessment
    - To the teacher
  - What does the teacher do with this information?
    - He/she informs the student how he fared compared to the standard / class - classification. (summative)
    - He/she informs the student how he can continue to learn or what progress he has made. (formative)
    - He/she adapts teaching. (formative)

Components of formative assessment

- Provided by a teacher.
- Self-assessment.
- Peer-assessment.

Types of formative assessment

- Proximal formative assessment
  - "here and now"
  - Present in verbal communication with pupils
    - Asking questions
    - Teacher's reaction to the correct answer
    - Teacher's reaction to an incorrect answer

        - The teacher's reflection to the student's initiative
    - Spontaneous, but you can prepare for it
- Formative assessment with an instrument
    - “Pre-prepared tool”
    - Specific formative assessment tools are present
        - Papers without marks
        - Clear communication of goals (rubrics, checklist, ...)
        - Activity / game
        - Prior assessment of understanding, pre-concepts
        - …
    - designed in advance

A good tool for formative assessment in mathematics addresses the following questions:

1. What do I want to formatively assess?
2. Is the tool mathematically correct?
3. What is its diagnostic potential?

1. What do I want to formatively assess?

- It is clear what the tool aims to assess formatively
    - Procedural knowledge
        - Timing:
            - At the beginning or before starting the unit: To identify level of entry of procedural knowledge (e.g. before construction tasks in geometry, identify students level of using geometric tools)
            - During the year: Automation (speed and preciseness) of some procedures (e.g. fluency in using multiplication table)
        - In which case:
            - Only assess what makes sense to process automatically (fast and precisely)
            - Only assess what is already understood
    - Conceptual knowledge
        - At the beginning of the unit:
            - Preconceptions (e.g. before linear function, students’ understanding of tables)
            - Understanding of prior knowledge
        - When using inquiry based learning:
            - It can serve as an indirect reference to knowledge which can be used when exploring the topic
        - At the end of the unit:
            - Level of understanding of the concept / solution method / ...
            - Possibility to verify propaedeutic goals as well

2. Is the tool mathematically correct?

- It is correct
    - Procedural knowledge
        - Is the tool mathematically correct?
        - Evaluates processes that need to be automated?
    - Conceptual knowledge
        - Is the tool mathematically correct?

3. **What is its diagnostic potential?**

- It has a good diagnostic potential
    - Procedural knowledge:
        - Does it allow you to evaluate progress? (related to speed or complexity)
    - Conceptual knowledge:
        - Does it allow the teacher to reveal pre-concepts or misconceptions?
        - Does it help to identify deep vs. superficial understanding of the concept, method?
        - Does it help to identify level of thinking, aspects used?
    - Does it inform teacher concerning the next teaching steps?

Activity 2. What exactly does a teacher do when formatively assessing?
Brainstorming.

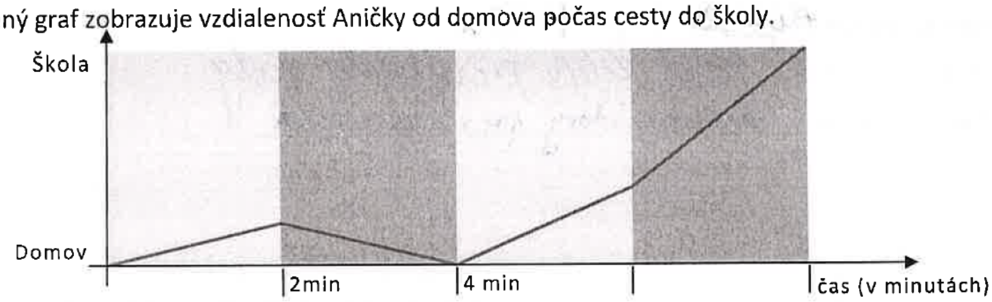
- Observes and identifies
    - levels of thinking
    - pre-concepts
    - misconceptions
- Finds out students' preferences (e.g. which representation is closer to them)
- Recognises what language pupils use
- Responds to support pupil learning
    - adapts the language
    - works with pupil error
    - shifts the responsibility for correctness to the pupil
- Ensures understanding for all pupils
- ...
- It is about moving from the role of "know-it-all" to the role of "facilitator"

Students' misconceptions (70 minutes)

Activity 3.
Analyse the following students' solutions. Assess their correctness, identify the representations and aspects of functional thinking that the student used.

Solution 1

Uvedený graf zobrazuje vzdialenosť Aničky od domova počas cesty do školy.



Popíšte Aničkin pohyb v oblastiach ①, ②, ③ a ④. V každej oblasti popíšte, akou rýchlosťou a akým smerom Anička išla. Použite na to napríklad vyjadrenia ako: „išla pomaly", „išla rýchlejšie ako", „išla smerom k škole", ...

vo fáza ~~jedna~~ išla rýchlejšie va fázy 2 pomalšie vo fázy 3 išla rýchlejšie a vo fázy 4 išla ešte rýchlejšie

(Source of the task: Sproesser et al., 2021)

- Representations: a graph to a verbal description.
- The student interprets the decrease of the graph of the given function as a slowdown, its growth as an acceleration, while he perceives the importance of the slope.
- Thus, we can see the use of the covariational approach which however, is not fully grasped.

Solution 2

Máme nasledujúcu situáciu:

„Taxislužba účtuje základný poplatok vo výške 2,50 € a za každý prejdený kilometer 0,80 €."

Ktorý z nasledujúcich predpisov správne opisuje situáciu? Napíšte, čo vyjadrujú premenné x a y.

☒ $y = 2{,}5x + 0{,}8$ ☐ $y = 2{,}5x - 0{,}8$ ☐ $y = 0{,}8x + 2{,}5$ ☐ $y = -0{,}8x + 2{,}5$

Premenná x vyjadruje: základný poplatok 2,5 €

Premenná y vyjadruje: ~~prej~~ poplatok navyše 0,8 €

(Source of the task: adapted from Nitsch, n.d.)

- Representations: a verbal description to a formula.
- The task evokes the use of the correspondence approach, but the student is not yet at this level of thinking. He could not identify the dependent and independent variable. In the

formula of the linear function, he put the numbers from the assignment in the order in which they appeared.

Solution 3

Máme nasledujúcu situáciu:

„Sviečka je na začiatku 24 cm vysoká a každú hodinu sa zmenší o 2 cm."

Ktorý z nasledujúcich predpisov správne opisuje situáciu? Napíšte, čo vyjadrujú premenné x a y.

☐ $y = 24x - 2$ ☐ $y = 2x + 24$ ☐ $y = -24x + 2$ ☒ $y = -2x + 24$

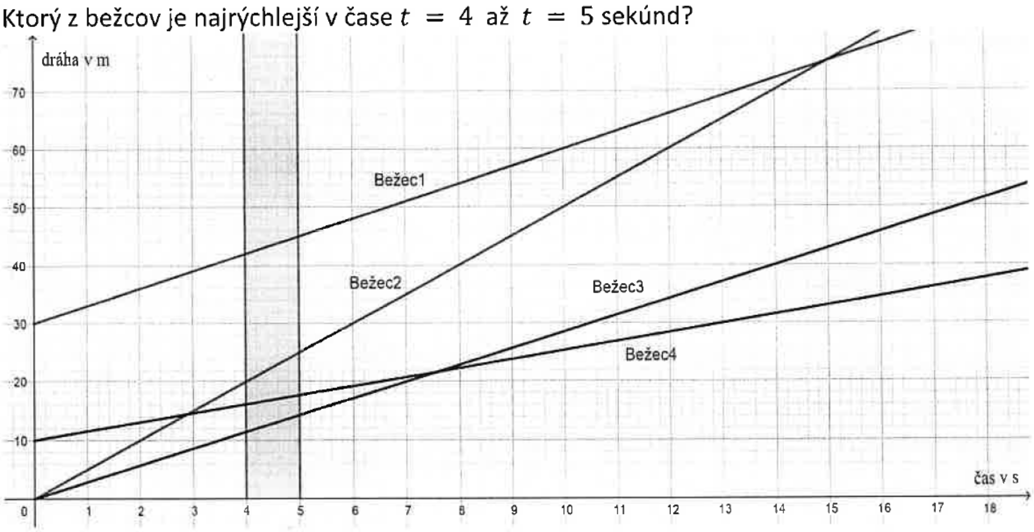
Premenná x vyjadruje: ~~hodiny~~ hodiny

Premenná y vyjadruje: stav sviečky

- Representations: a verbal description to a formula.
- The task evokes the use of the correspondence approach. In contrast to student solution 2, the student correctly identifies the individual variables. We can see that he first struggles with a similar misconception as he takes the order of values from entering into the general formula of a linear equation. Then he makes a correction to the correct solution.

Solution 4

Ktorý z bežcov je najrýchlejší v čase $t = 4$ až $t = 5$ sekúnd?



☐ Bežec 1 ☒ Bežec 2 ☐ Bežec 3 ☐ Bežec 4

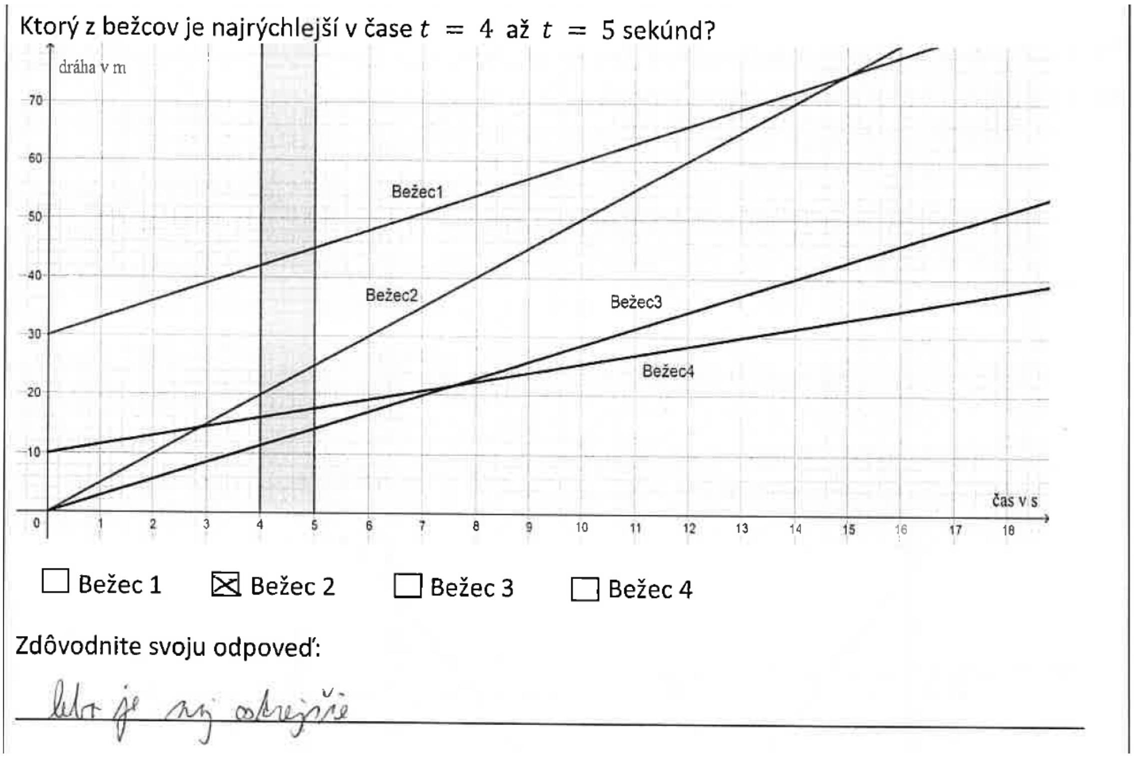
Zdôvodnite svoju odpoveď:

Čiarka je nakreslená vyššie. Akože viac do hore.

(Source of the task: adapted from Nitsch, 2015)

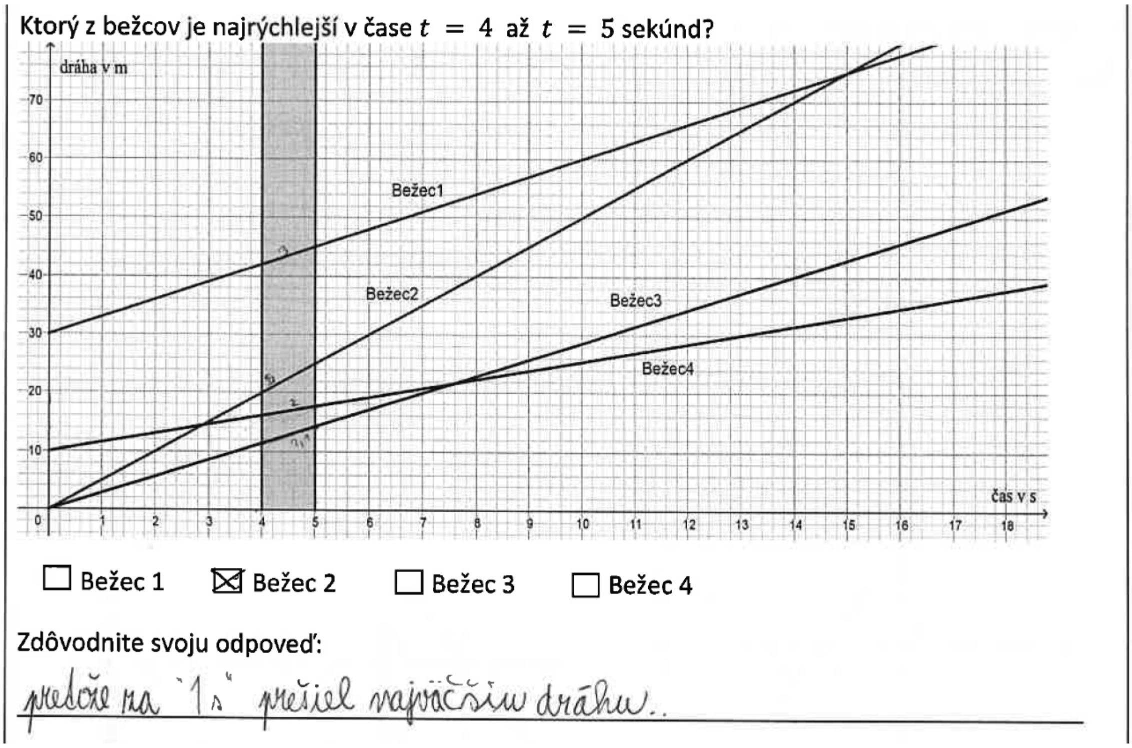
- This task involves understanding the graphical representation of a function.
- The student solves the task correctly - he identifies runner 2 as the fastest, but does not know the correct terminology to name the largest directive slope? of this linear function. That's why he used the term "the line goes further up". We can interpret his focus on the slope as the use of the covariatioanl approach.

Solution 5



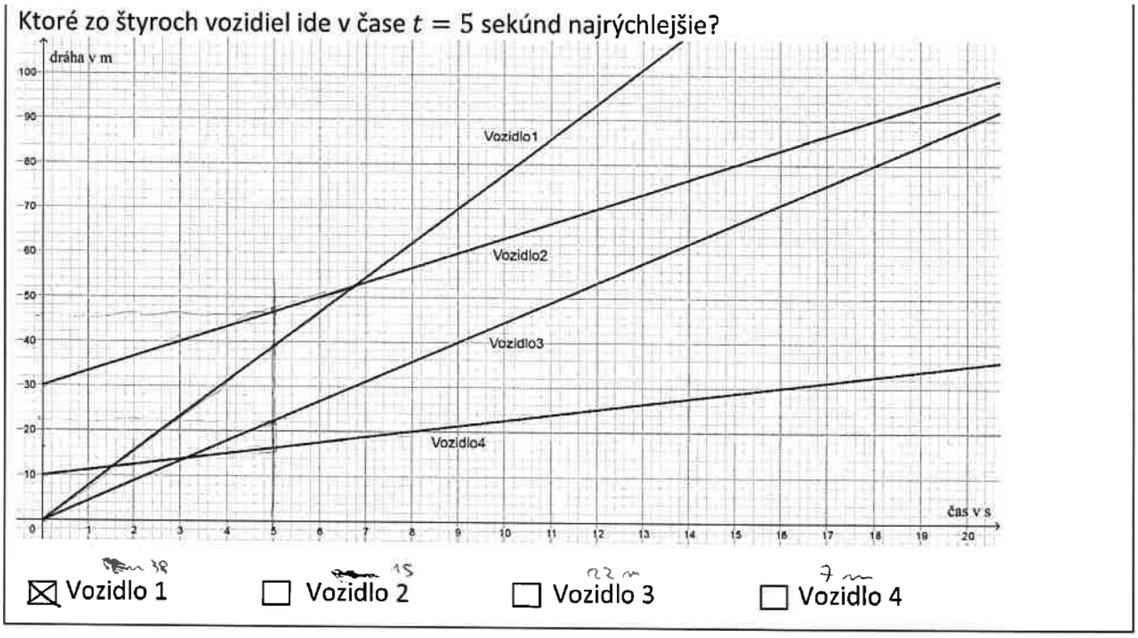
- This solution is analogous to the previous one. The visual perception of the largest guideline among the graphs offered, this time the student named it "the sharpest"

Solution 6



- The student solves the task correctly - he identifies runner 2 as the fastest. It can be seen that he focused on the speed of as? change - how many meters did the individual runners move in 1s (from 4 to 5 seconds).

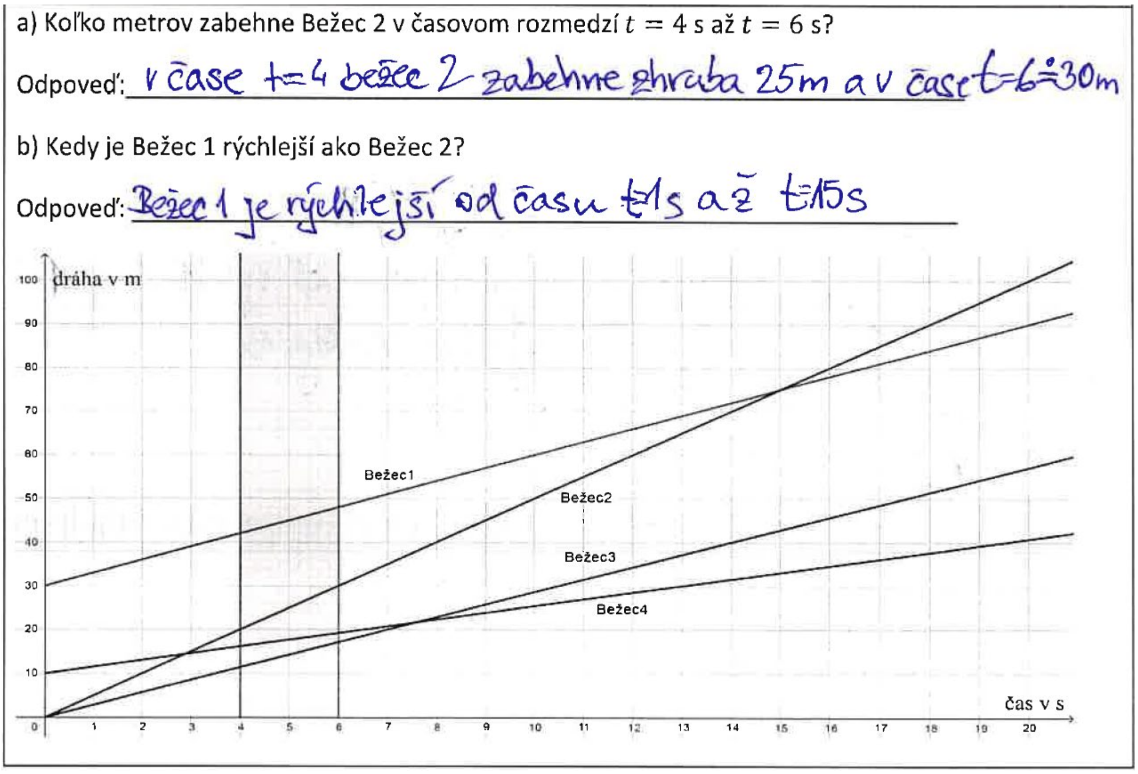
Solution 7



(Source of the task: adapted from Nitsch, 2015)

- This task involves understanding the graphical representation of a function.
- The student solves the task correctly - identifies vehicle 1 as the fastest, counts how many meters each vehicle traveled in the first 5 seconds. It is not clear whether the student perceives the movement is uniform.

Solution 8

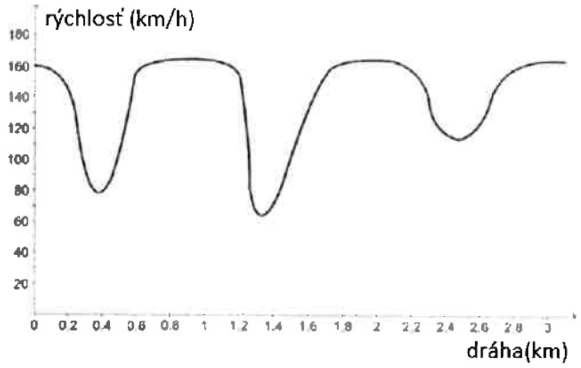


- This task involves understanding the graphical representation of a function.

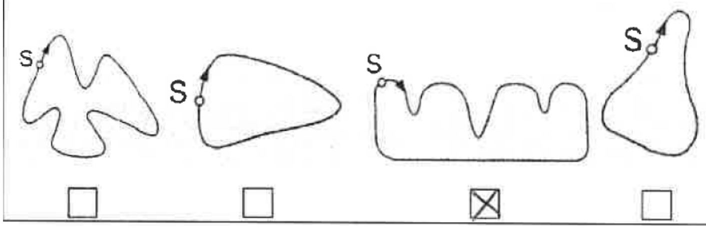
- The student solves the problem incorrectly - he does not focus on the change of track, but only on the functional values of the individual time points. So either he misread the description on the *y*-axis, or he wants to solve the task at the input-output level, which is not sufficient for this task.

Solution 9

Tento graf znázorňuje, ako sa mení rýchlosť pretekárskeho auta na dlhom rovinatom pretekárskom okruhu počas druhého kola pretekov.



Nasledujúce obrázky sú náčrtmi štyroch pretekárskych okruhov. Na ktorom z týchto okruhov jazdilo pretekárske auto, ktorého graf rýchlosti je načrtnutý vyššie?

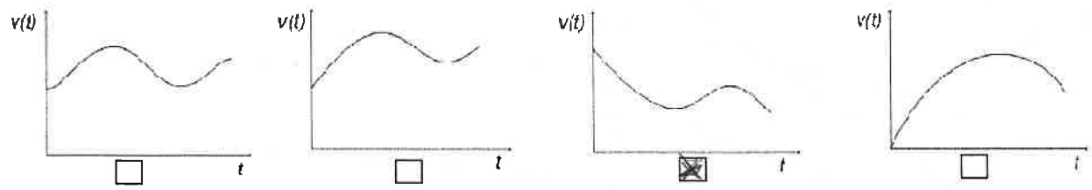


(Source of the task: adapted from Nitsch, 2015)

Solution 10

Nasledujúci obrázok znázorňuje lyžiara jazdiaceho po svahu.
Hodnota funkcie $v(t)$ udáva jeho okamžitú rýchlosť v čase t.

Ktorý z grafov najlepšie opisuje uvedenú situáciu?



Vysvetli svoju odpoveď:

(Source of the task: adapted from Nitsch, 2015)

Solution 11

Do vázy na kvety, ktorá je znázornená na obrázku, napúšťame vodu rovnomerným prítokom vody. Načrtnite graf funkcie, ktorá vyjadruje výšku hladiny vody vo váze v závislosti od času.

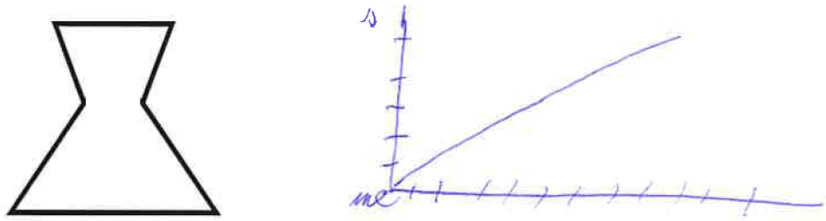
Solution 12

Do vázy na kvety, ktorá je znázornená na obrázku, napúšťame vodu rovnomerným prítokom vody. Načrtnite graf funkcie, ktorá vyjadruje výšku hladiny vody vo váze v závislosti od času.

The previous problems (9-12) point out a common problem when sketching a graph. The student looks at the situation holistically (apparently using the covarianal approach), but lacks insight, uses a shortcut solution, specifically draws the function as a picture.

Solution 13

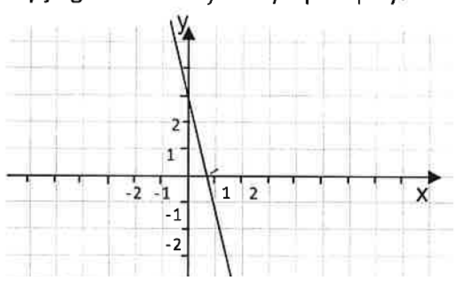
Do vázy na kvety, ktorá je znázornená na obrázku, napúšťame vodu rovnomerným prítokom vody. Načrtnite graf funkcie, ktorá vyjadruje výšku hladiny vody vo váze v závislosti od času.



In this task, the student focused on the term "uniform increase", from which he concluded that it would be a linear function. This is an incorrect use of the covarianal approach.

Solution 14

Daný je graf funkcie f a štyri predpisy.

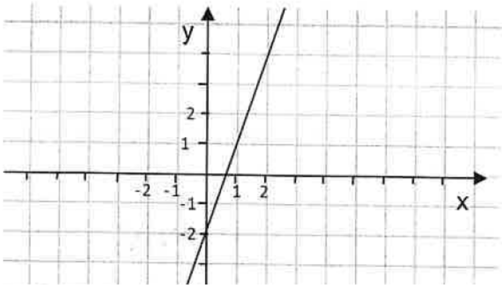


Ktorý predpis zodpovedá grafu funkcie f?

- $y = -4x + 3$
- $y = 3x + 0{,}75$
- X $y = 0{,}75x + 3$
- $y = 3x - 4$

Solution 15

Daný je graf funkcie:



Zapíšte predpis funkcie zodpovedajúcej grafu.

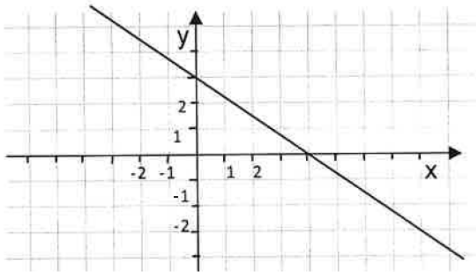
$y = -2x + 0{,}4$

Stručne vysvetlite, ako ste postupovali.

podľa súradníc

Solution 16

Daný je graf funkcie f:



Zapíšte predpis funkcie zodpovedajúcej grafu.

$y = 4x+3$

Stručne vysvetlite, ako ste postupovali.

y=[0;3] x=[0;4]

podľa súradníc

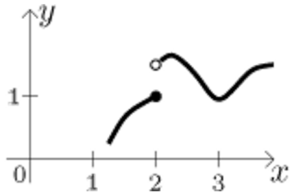
The previous problems (14-16) show a standard misconception where the y-intercept isinterchanged with the x-intercept.

Activity 4.
In small groups, write as long of a list as possible with misconceptions that we might encounter specifically when teaching functions.

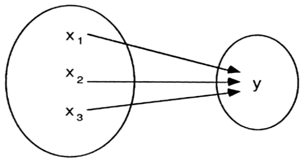
Pre-service teachers work on the topic in small groups. They can use knowledge from the previous activity. Afterwards, they collect their ideas and the lector presents the following “list of misconceptions, errors and difficulties.” Its structure might be overlapping in some cases:

1. **What is function?**
   - inaccurate ideas about what graphs of functions should look like (only graphs that show an obvious or simple pattern are understood as graphs of functions)
   - the idea that only graphs where a pattern is apparent represent functions; others look strange, artificial or unnatural.
   - the idea that the following functions are not functions: functions composed of arbitrary correspondences, functions given by more than one rule, and functions that are not officially recognized and labeled as functions by mathematicians
   - the idea that functions must consist of quantities which are variable
   - the idea that a function implies causality
   - the belief that functions are always simple, the chaos between assigning "one to many" and "many to one"
   - **Examples:**

The graph does not represent a function because it does not exhibit an obvious pattern, a regularity.



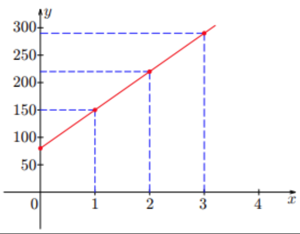
This is not a function because several *x* points lead to one *y*.



2. Linearity

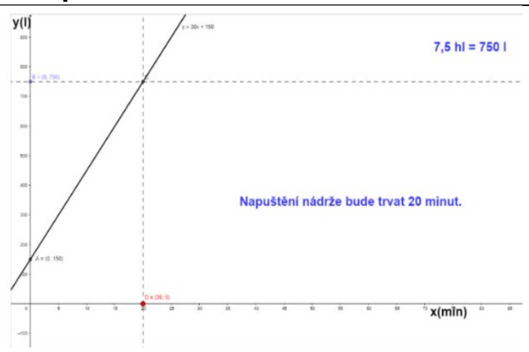
- the tendency to define a function as a relationship that, when represented graphically, produces a linear pattern
- the tendency to connect every two consecutive points with a straight line (in both contextual and abstract situations)
- only one function can pass through two given points (a generalization of a special property of linear functions)
- over-generalisation of properties of linear functions to other types of functions
- **Example:**

Alex bought a new car with an odometer reading 80 km. However, that is about to change as he is going on a bigger trip tomorrow. Describe how you can represent (table, graph, equation) that his average speed is 70 km per hour.



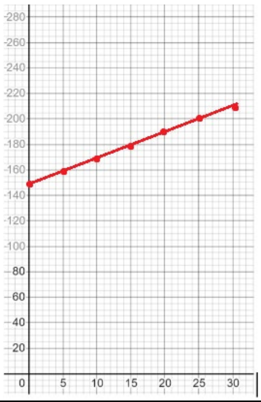
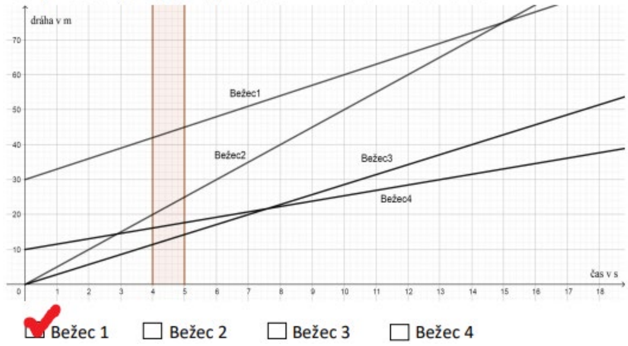
3. Domain and range

- confusion of the value domain and the definition domain
- misunderstanding how the definitional scope affects the value scope
- ignoring the definitional scope in contextual tasks
- **Example:**



Translation: Filling the tank will take 20 minutes.

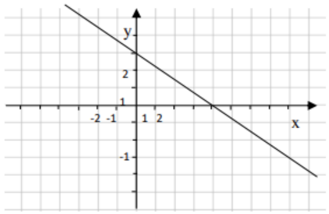
4. **Difficulties with graphical representation and coordinate system**
    - representation or interpretation of continuous data in a discrete way or vice versa
    - interval and point substitution
    - confusion of slope and height
    - a graph as an image
    - difficulties in setting two axes for a Cartesian coordinate system
    - scaling problems
    - the effect of changing the scale of the axes on the appearance of the graph
    - confusion of two axes of a graph
    - misunderstanding the meaning of points in the same position relative to one of the axes
    - points on the graph remain in the same position even if the axes change
    - graphs always pass through (or start at) the origin
    - the largest numbers marked on the axes represent the largest values reached
    - **Examples:**

Graph the dependence of the volume in the measuring cup and the number of beads. 
Ktorý z bežcov je najrýchlejší v čase $t = 4$ až $t = 5$ sekúnd?  *Translation: Which runner is the fastest in the time t=4s to t=5s?*

5. **Difficulties with formula and variables**
    - misunderstanding the difference between coefficients and variables (visible when fitting point coordinates to the general form of a given function)
    - algebraic problems when modifying a prescription
    - changing the symbol of a variable in a functional equation changes some critical aspects of the function
    - misunderstanding the meaning of coefficients (e.g., directives)
    - for a linear function:
        - intercepts as coefficients

- the order of values in the verbal description affects the order of coefficients in the prescription
- confusion between the coefficients k and q

- **Examples:**

The taxi service charges a basic fee of €2.50 and €0.80 per kilometer travelled. $y = 2{,}5x + 0{,}8$
Daný je graf funkcie f:  $y = 4x + 3$

6. **Lack of understanding of different aspects**
   - **Input Output**
     - e.g. problems when filling in a table and reading data from it
   - **Covariance**
     - e.g. ignoring covariance properties of individual functions
   - **Correspondence**
     - e.g. inability to generalise a relationship using a prescription
   - **Object**
     - e.g. difficulty in understanding the properties of functions
     - e.g. failure to identify the type of a function by its prescription if the prescription is not in its basic form

Lesson Plan Teacher Course

Formative assessment – Reaction on incorrect answer	
Teaching hours:	45 minutes
Target group:	Pre-service teachers
Important links	
Videos	Marbles 2 IV Marbles 3 IV https://www.funthink.eu/learning-environments/lower-secondary-education/marbles (after login)
Description	
Goals:	• Pre-service teachers realize different strategies connected with their possible reaction on the incorrect answer. • They are able to reflect on the consequences of different teachers' strategies on students (functional) thinking development. • They start to reflect on their own tendencies when reacting on the incorrect answer.
Structure:	• 45 minutes Role-play on different reactions on students' incorrect answers, solution • 45 minutes Video-analyses of teachers' reactions on students incorrect solutions

Activities

Role-play (45 minutes)

Activity 1. How could this happen in the classroom?
You will be randomly assigned one of the situations. Prepare a short role-play on how could it look in the classroom. Please, develop the details of the scene.

- Firstly, revise all together the formative assessment skills that are necessary for teachers. Pinpoint, that this session is going to focus on one of them: Reaction on the incorrect answer.
- A lector forms groups of three and randomly assign them one or two situations of the following. The group keeps the assignment as a secret:
  - Student answers incorrectly / provides incorrect solution and teacher asks another student to answer correctly
  - Student answers incorrectly / provides incorrect solution and teacher tells correct answer instead of student
  - Student answers incorrectly / provides incorrect solution and teacher tells it is incorrect and asks the student another questions Student answers incorrectly / provides incorrect solution and teacher asks the class to evaluate the answer
  - Student answers incorrectly / provides incorrect solution and teacher starts a discussion about this and other (correct) solutions of another student
  - Student answers incorrectly / provides incorrect solution and teacher is asking questions so that the student starts to be confused about his / her own response
- A lector encourages students to develop also some details of a situation (what was incorrectly said)
- Each group role-plays their situation, other PSTs are discussing what kind of reaction it was
- All groups together generate other ideas how the teacher can respond on an incorrect answer/solution.

Video-analysis (45 minutes)

Activity 2. What can you see in the video?
After each video:

A. Write down what you have seen that happened as precisely as possible. Do this on your own.
B. Interpret what teacher and student(s) were thinking and think about possible consequences (if this was the most common teacher's reaction on students' error).

The videos, their descriptions and the subtitles are to be found here:

Videos: https://www.funthink.eu/learning-environments/lower-secondary-education/marbles
(after login)
Descriptions and subtitles: handout document

Before each of the videos, introduce a context necessary for video understanding.
Especially, make clear what task is discussed.

After the individual and group work, facilitate the whole group discussion.