Co-funded by the
Erasmus+ Programme
of the European Union

Knowledge of topic

Name of the course

Name of the Lector

Session number:

Intellectual output of FunThink (Erasmus+) project

Activity 1:
Lift

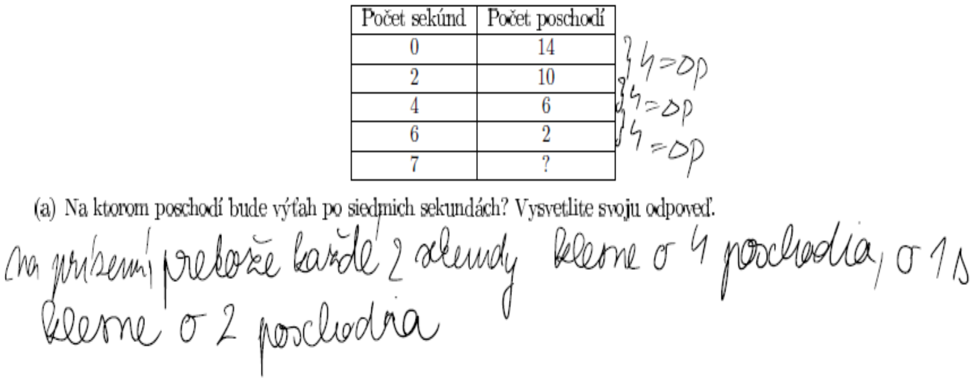
The hotel has several floors, the ground floor with number zero, and below the ground floor there are several parking floors. The following table shows which floor you can reach after a few seconds. You have checked in on the 14th floor and are about to take the lift down to the parking lot to your car.

Počet sekúnd	Počet poschodí
0	14
2	10
4	6
6	2
7	?

A. Which floor will the lift be on after seven seconds? Explain your answer.

B. At what speed is the lift descending? Explain your answer.

Activity 1:
Lift

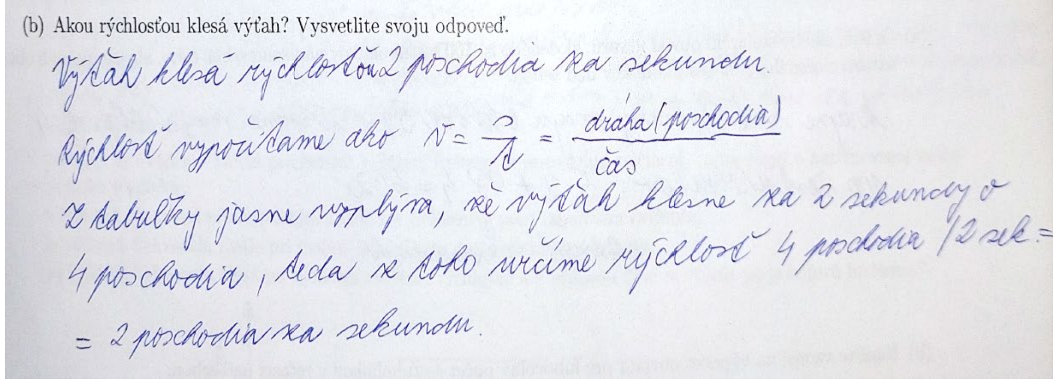


Počet sekúnd	Počet poschodí
0	14
2	10
4	6
6	2
7	?

}4 => op
}4 => op
}4 => op

(a) Na ktorom poschodí bude výťah po siedmich sekundách? Vysvetlite svoju odpoveď.

Na prízemí, pretože každé 2 sekundy klesne o 4 poschodia, o 1 s klesne o 2 poschodia

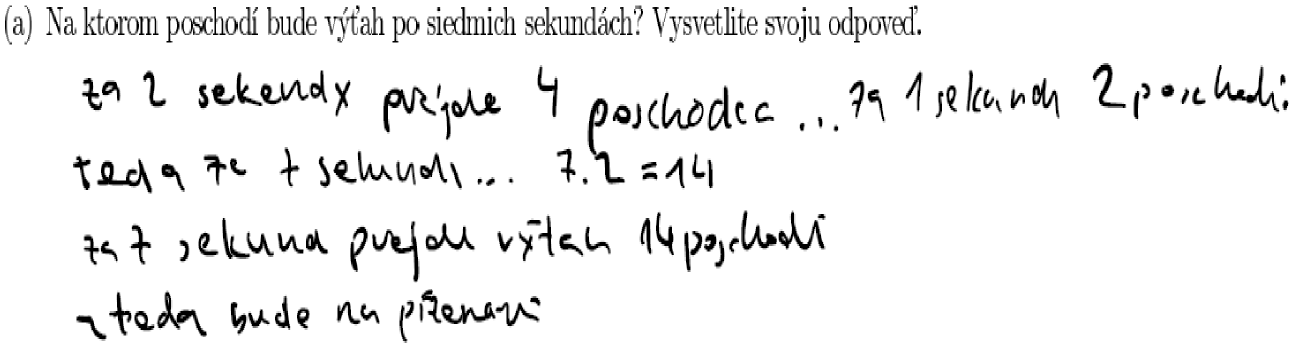


(b) Akou rýchlosťou klesá výťah? Vysvetlite svoju odpoveď.

Výťah klesá rýchlosťou 2 poschodia za sekundu.

Rýchlosť vypočítame ako $v = \frac{s}{t} = \frac{\text{dráha (poschodia)}}{\text{čas}}$

Z tabuľky jasne vyplýva, že výťah klesne za 2 sekundy o 4 poschodia, teda z toho určíme rýchlosť 4 poschodia / 2 sek = = 2 poschodia za sekundu.



(a) Na ktorom poschodí bude výťah po siedmich sekundách? Vysvetlite svoju odpoveď.

za 2 sekundy prejde 4 poschodia ... za 1 sekundu 2 poschodia

teda za 7 sekundy ... 7.2 = 14

za 7 sekund prejde výťah 14 poschodí

→ teda bude na prízemí

s = vt

poítame rýchlos pre 10. poschodie:
naše s = koko poschodí
naše v = neznáma
naše t = 2 sekundy (výah sa po 2s dostane na 10 poschodie)

doplníme do vzorca: 4 = v . 2 → 1 = v . 0,5
2 = v   2m/s = v
2 m/s = v

riešenie: výah sa pohybuje rýchlosou 2 m/s.

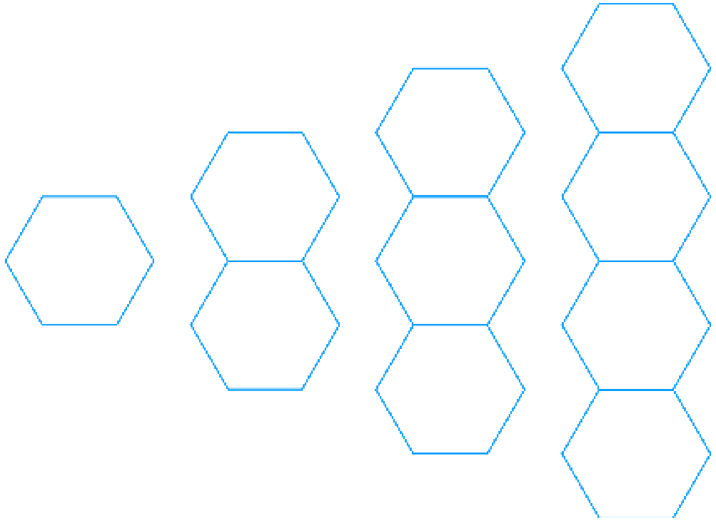
Activity 2:
Hexagons

The first figure (1 hexagon) has a perimeter of 6.

The third figure (3 hexagons on top of each other) has a perimeter of 14.

The second figure (2 hexagons on top of each other) has perimeter ____.

The fifth figure (5 hexagons on top of each other) has perimeter _____.

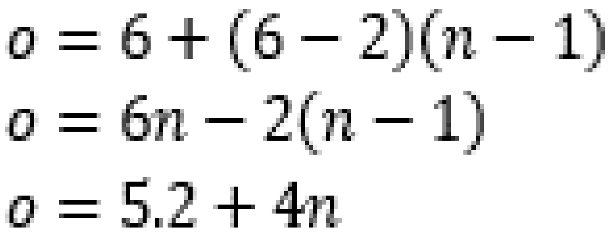


A. Describe how you would determine the perimeter of a figure composed of 100 hexagons on top of each other without knowing the perimeter of a figure composed of 99 hexagons on top of each other.

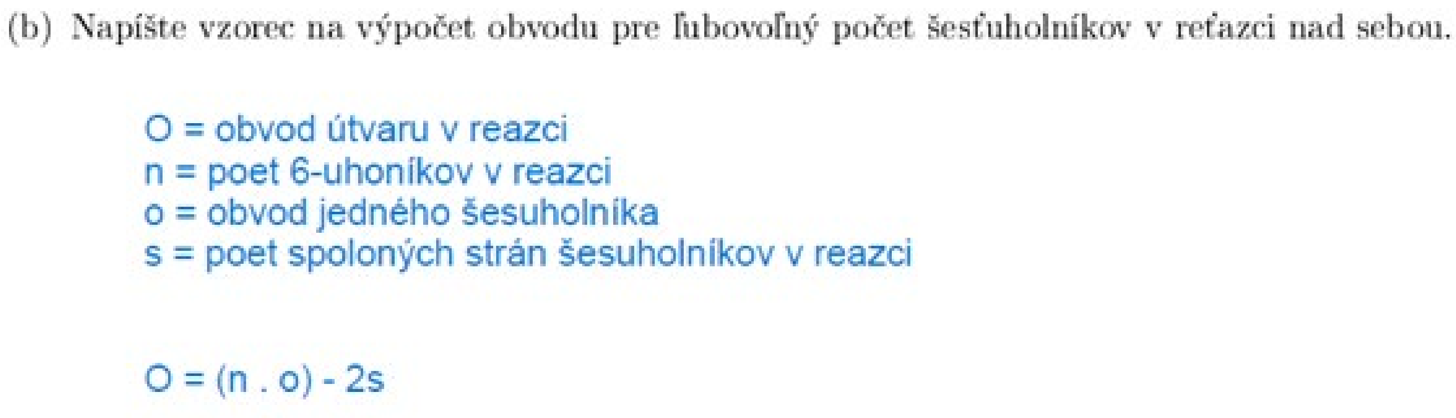
B. Write a formula to calculate the perimeter for any number of hexagons in a chain above.

C. Explain why your formula should be correct.

Activity 2:
Hexagons



$$o = 6 + (6 - 2)(n - 1)$$
$$o = 6n - 2(n - 1)$$
$$o = 5.2 + 4n$$



(b) Napíšte vzorec na výpočet obvodu pre ľubovoľný počet šesťuholníkov v reťazci nad sebou.

O = obvod útvaru v reazci
n = poet 6-uhoníkov v reazci
o = obvod jedného šesuholníka
s = poet spoloných strán šesuholníkov v reazci

O = (n . o) - 2s

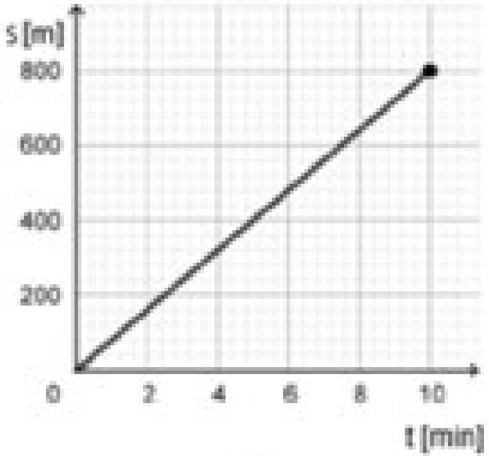
(c) Vysvetlite prečo by mal byť váš vzorec správny.

Poda ma je môj vzorec jednoducho pochopitený a jasný.

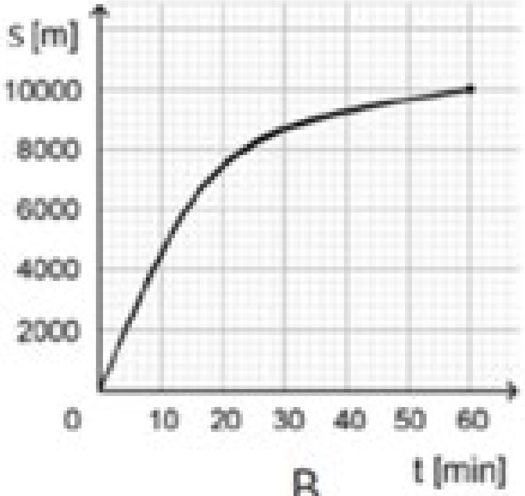
Activity 3:
Adam is running

Adam started from point **A** and ran 10 km (see picture).

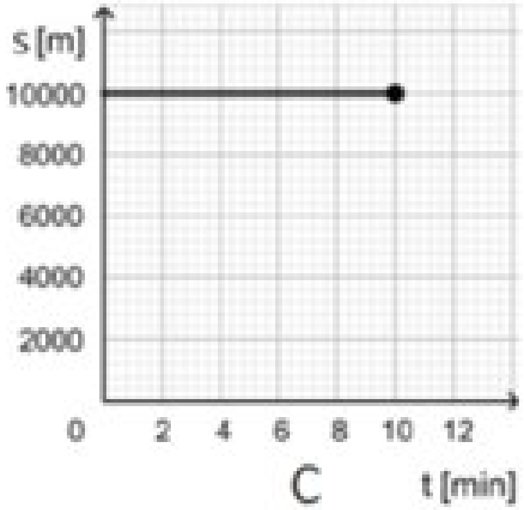
Choose a graph (circle the letter) that can describe its run and justify your choice.



A



B



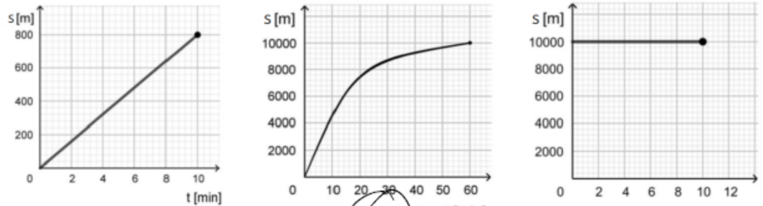
C

None of the options is correct.

D

Note: **S** denotes the distance of Adam from point **A** (in meters) and **t** denotes the time (in minutes).

Activity 3:
Adam is running

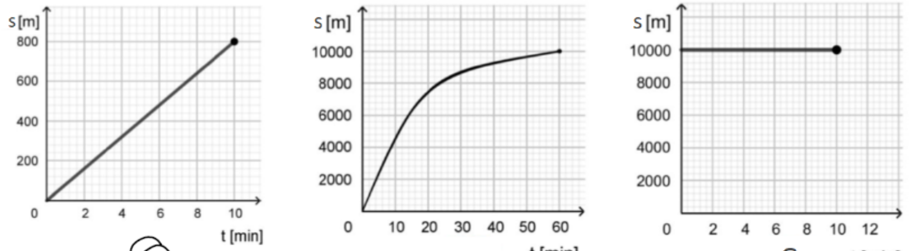


Žiadna z možností nie je správna.

A [✓ B] C D

Svoj výber vysvetlite: v prvej časti trate bežal rovnomerným pohybom väčšou rýchlosťou (strmší graf) a v poslednej časti už vládal menej, graf je menej strmý (nižšia rýchlosť), čas je 1h (realistickej na 10 km)

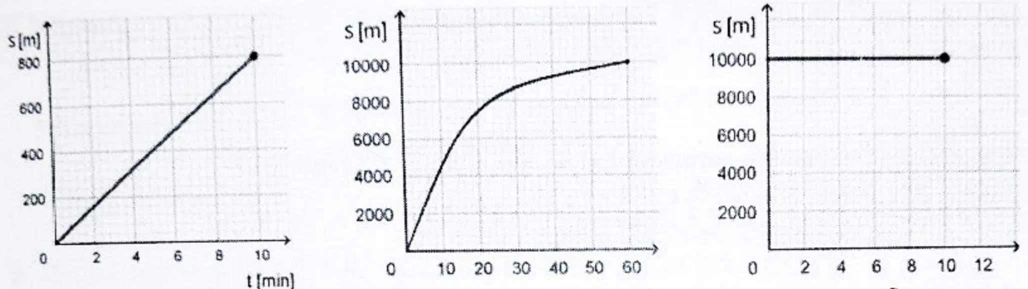
4.



Žiadna z možností nie je správna.

[✓ A] B C D

Svoj výber vysvetlite:
C to nemôže byť pretože by od 0 minúty mal prejdetých 10 km
môže to byť B al. A
B - vyjadruje že na začiatku bežal rýchlo a potom spomalil
A - vyjadruje že bežal konšt. rých. 4.



Žiadna z možností nie je správna.

A B C [✓ D]

Svoj výber vysvetlite: Adam počas behu určite nemal stále rovnaké tempo, spočiatku vládal viac, boli úseky, kedy nevládal bežať tak ako na začiatku, mohol mať prestávku na vodu, chvíľu mohol chodiť a následne zase bežať. Ak by sme vybrali A alebo C znamenalo by, že mal adam celý beh rovnaké tempo, čo musíme vylúčiť. B mi tak isto nepríde vhodné, za 10min adam zabehne viac ako 4 km, čo je veľa km za krátky čas

4.

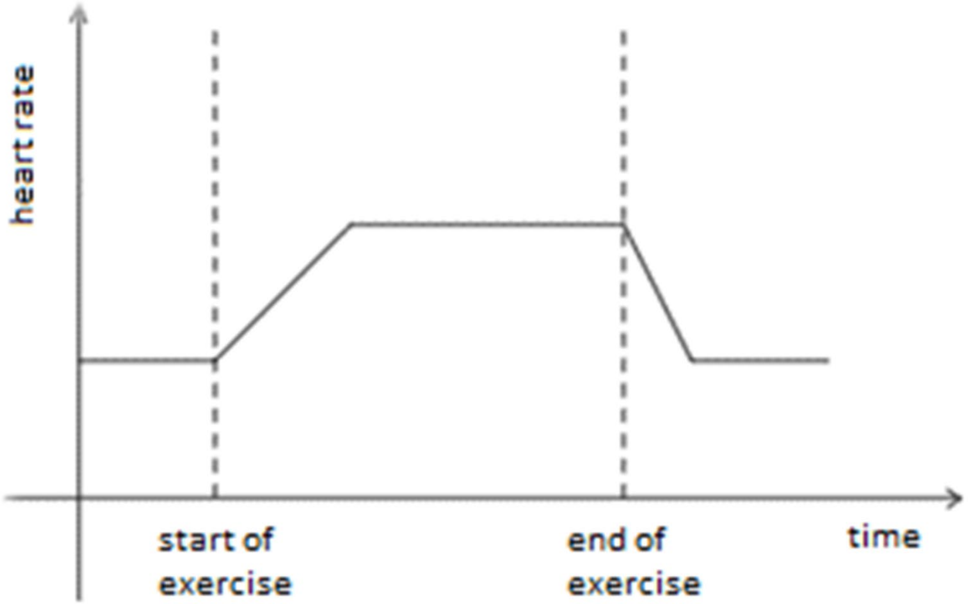
Activity 4:
Without training

The textbook states,

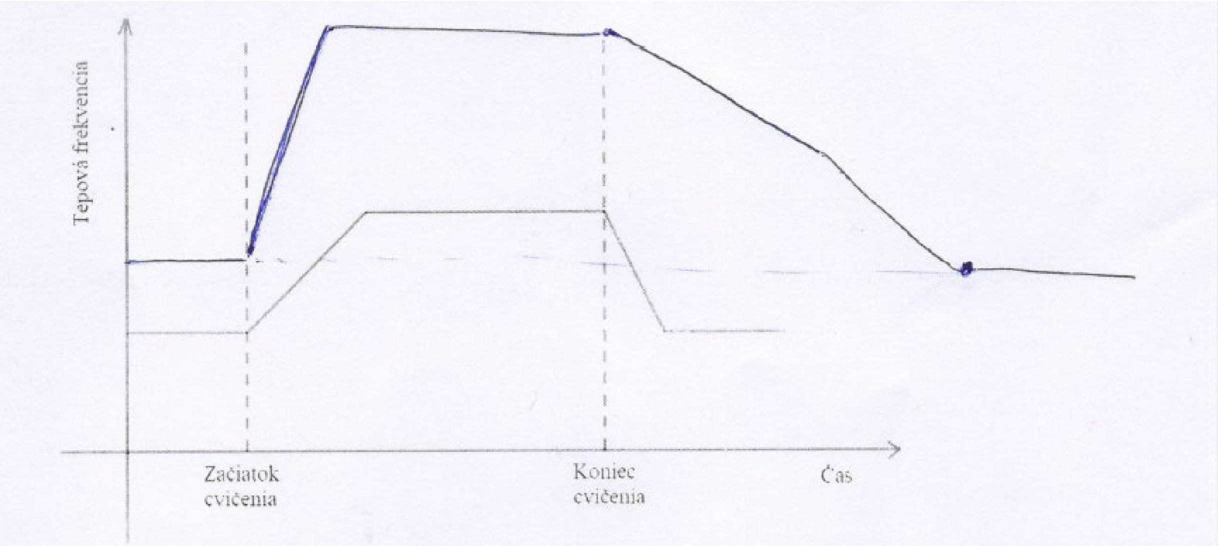
"There are several differences between the heart rate waveform of a regularly exercising - trained and untrained person:

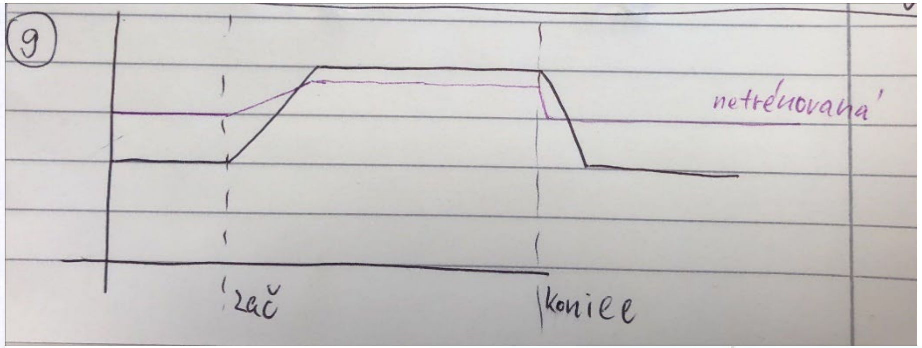
- the trained person has a lower resting heart rate before the start of the exercise,
- her heart rate rises more slowly with exercise and reaches lower values,
- her heart rate drops faster after exercise and returns to resting value in a shorter time."

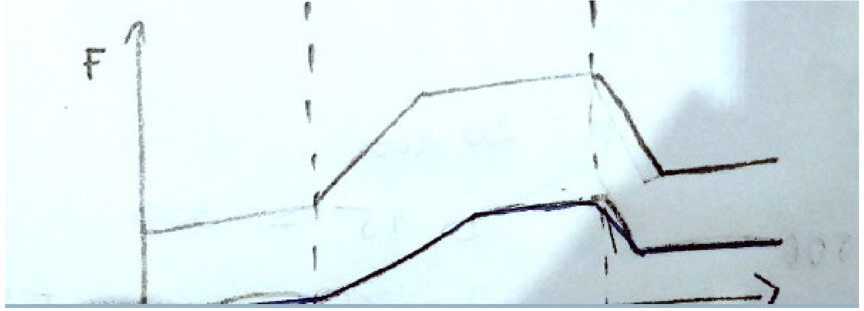
In the figure is a graph of the heart rate of a trained person. In the same figure, sketch what the heart rate graph of an untrained person would look like for the same exercise, satisfying all the above differences.

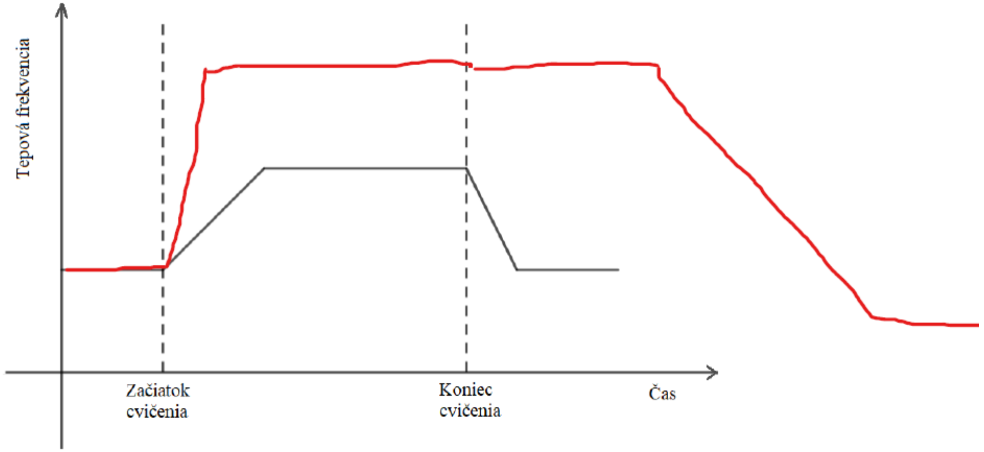


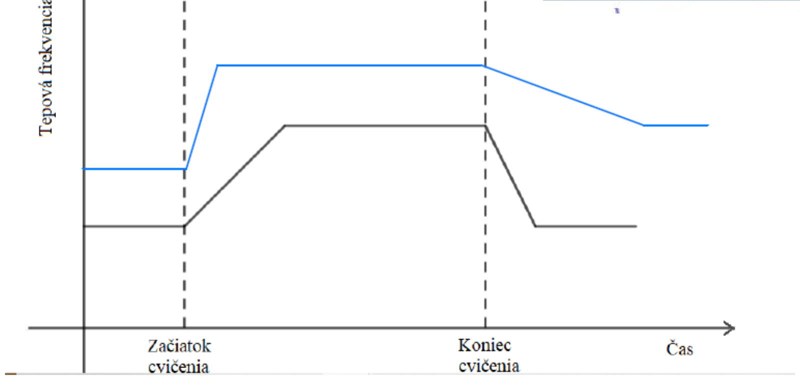
Activity 4:
Without training











Activity 5:
Runners

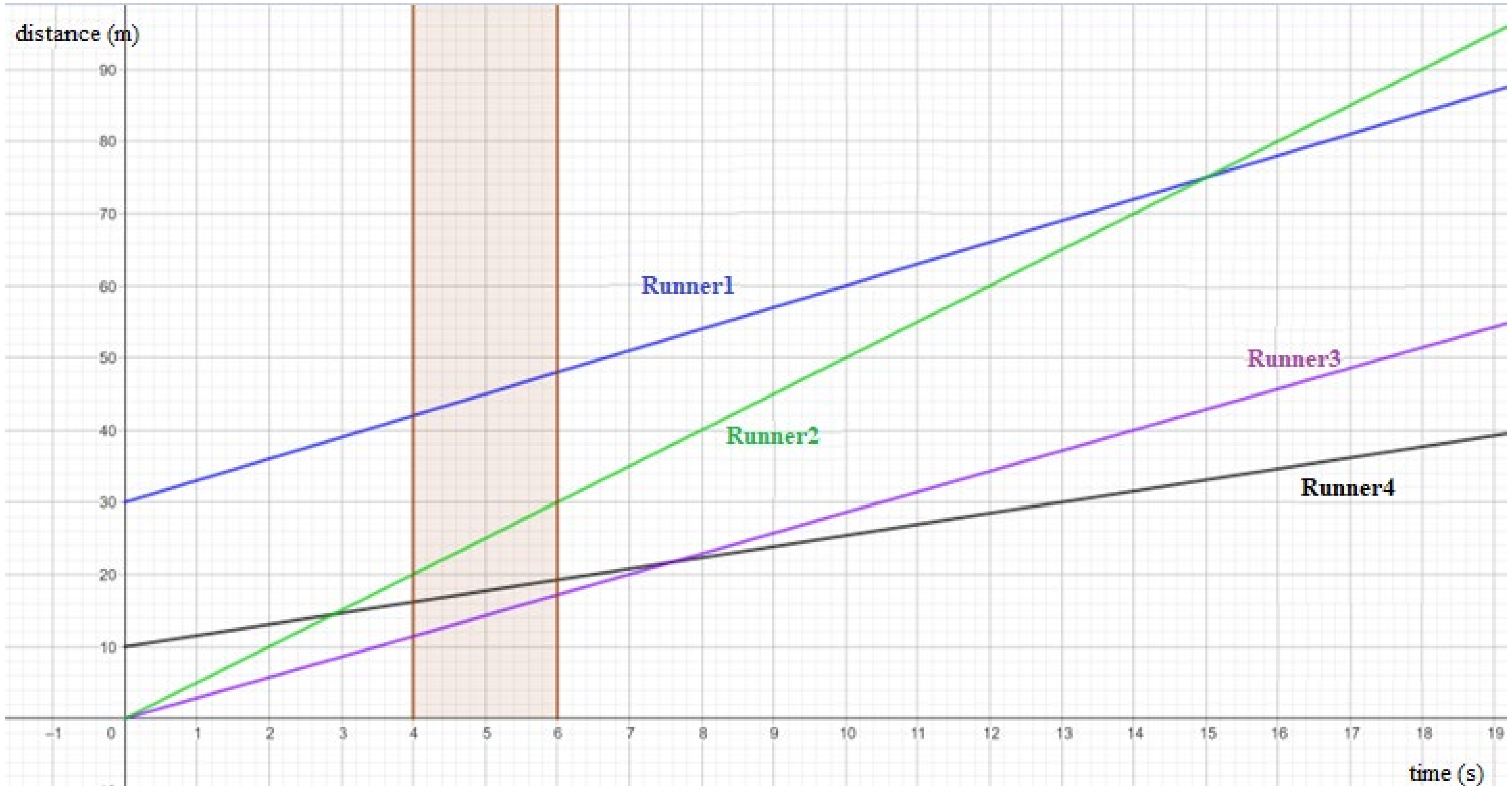
Find out the following information from the figure:

A. How many meters will **Runner 2** run in the time range t = 4s to t = 6s?

B. When is **Runner 1** faster than **Runner 2**?

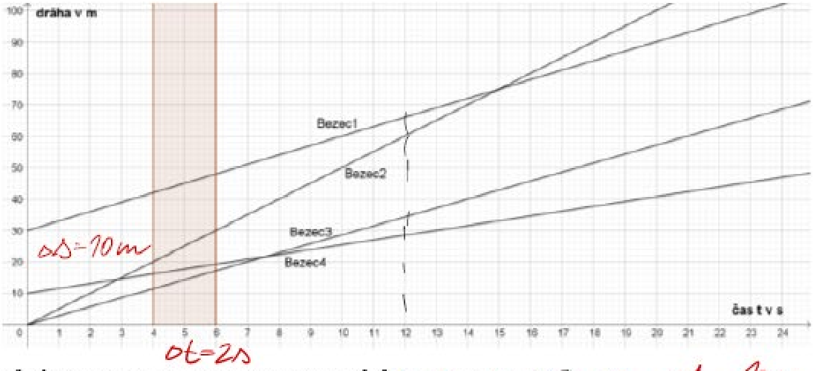
C. C. Which runner is the fastest at time t = 12s?

Give reasons for your answers.



Activity 5:
Runners

Z nasledujúceho obrázku zistite



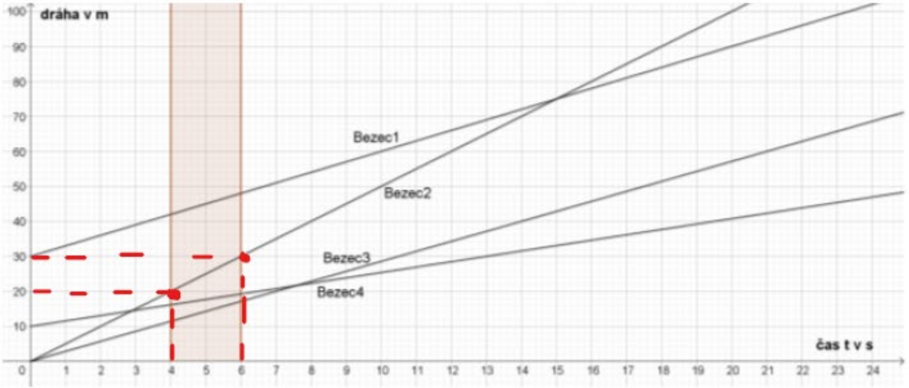
(a) Kolko metrov zabehne *Bežec 2* v časovom rozmedzí $t = 4s$ až $t = 6s$? $v_2 = \frac{\Delta s}{\Delta t} = \frac{10m}{2s} = 5m/s$

(b) Kedy je *Bežec 1* rýchlejší ako *Bežec 2*? nikdy (graf B1 je vždy menej strmý ako B2 → vždy je p.

(c) Ktorý bežec je v čase $t = 12s$ najrýchlejší?

bežec 2 → najstrmší graf

Z nasledujúceho obrázku zistite



(a) Koľko metrov zabehne *Bežec 2* v časovom rozmedzí $t = 4s$ až $t = 6s$? 10 metrov

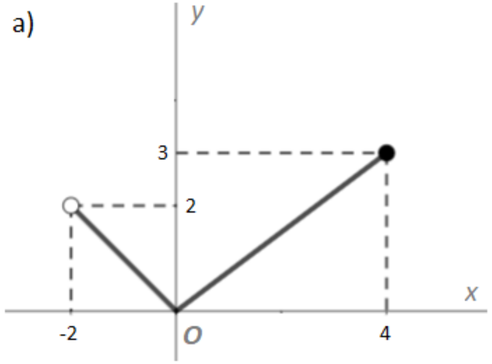
(b) Kedy je *Bežec 1* rýchlejší ako *Bežec 2*? od zaiatku, bežec 2 ho dobehne 15 sekunde

(c) Ktorý bežec je v čase $t = 12s$ najrýchlejší? bežec 1

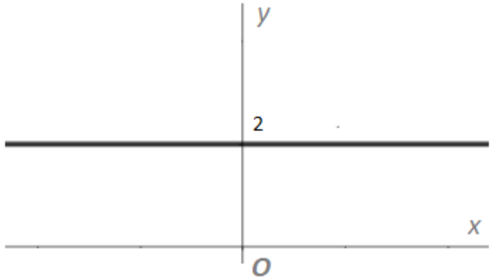
Activity 6:
Graph of a function

Decide which of the graphs shown in the following figures is the graph of a function. If possible, determine the defining domain and the range of the function.

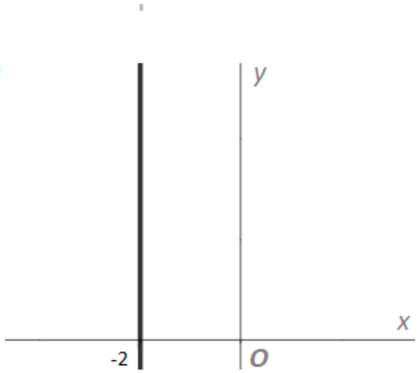
a)



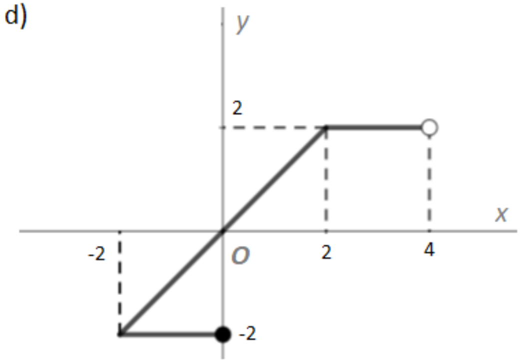
b)



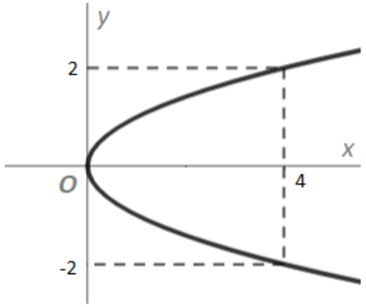
c)



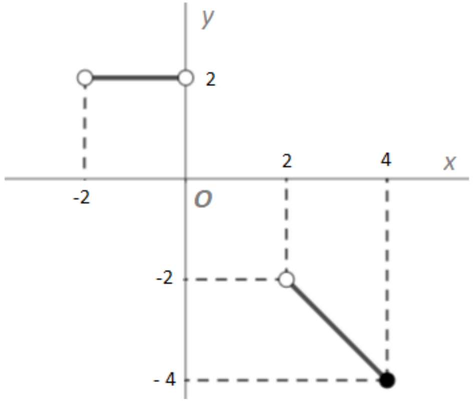
d)



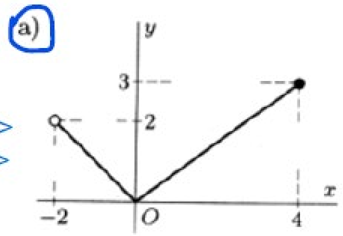
e)



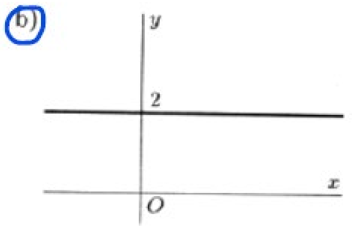
f)



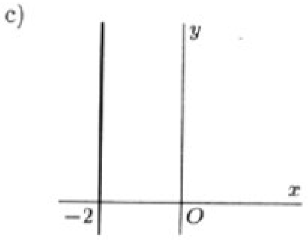
Activity 6:
Graph of a function

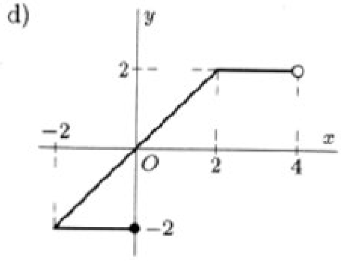


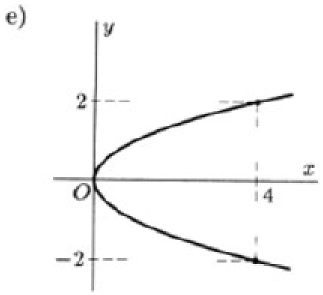
D(a)=(-2;4>
H(a)=<0;3>

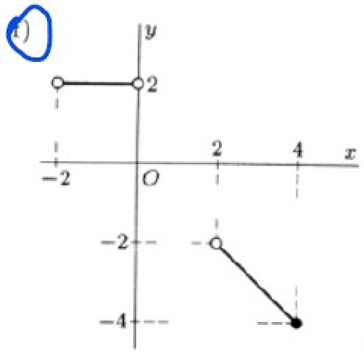


D(b)=R
H(b)={2}









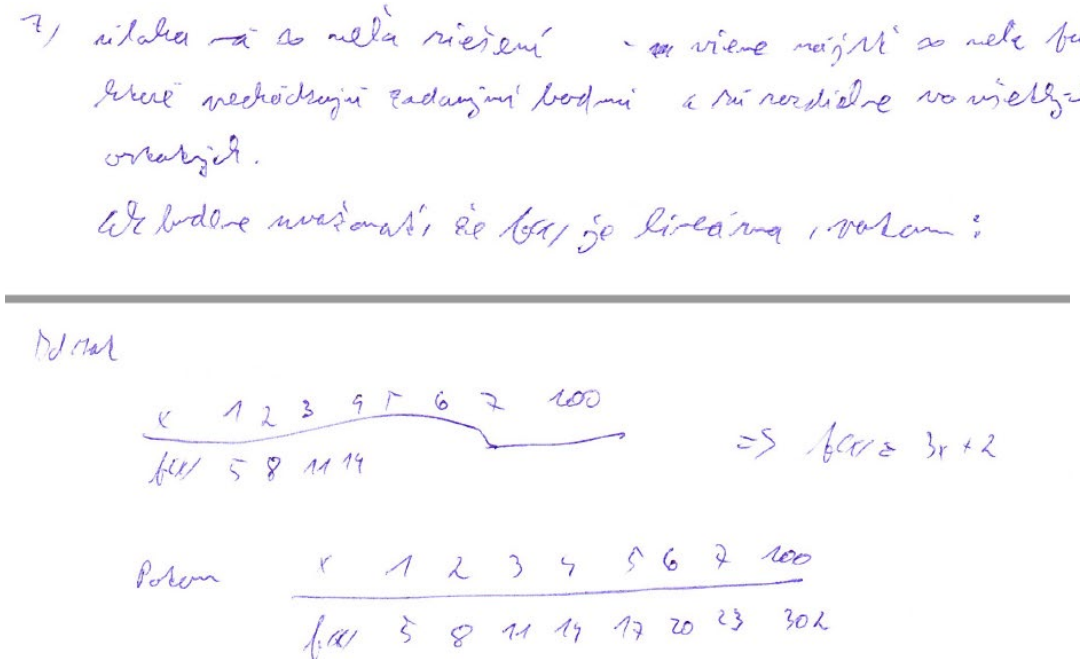
D(f)= (-2;0)U(2;4>
H(f)= <-4,-2)U{2}

Activity 7:
Formula from the table

Continue to fill in the table and find the formula of a function:

x	1	2	100
$f(x)$	5	8	

Activity 7:
Formula from the table



7/ úloha má ∞ veľa riešení – vieme nájsť ∞ veľa funkcií, ktoré prechádzajú zadanými bodmi a sú rozdielne vo všetkých ostatných.

Ak budeme uvažovať, že f(x) je lineárna, potom:

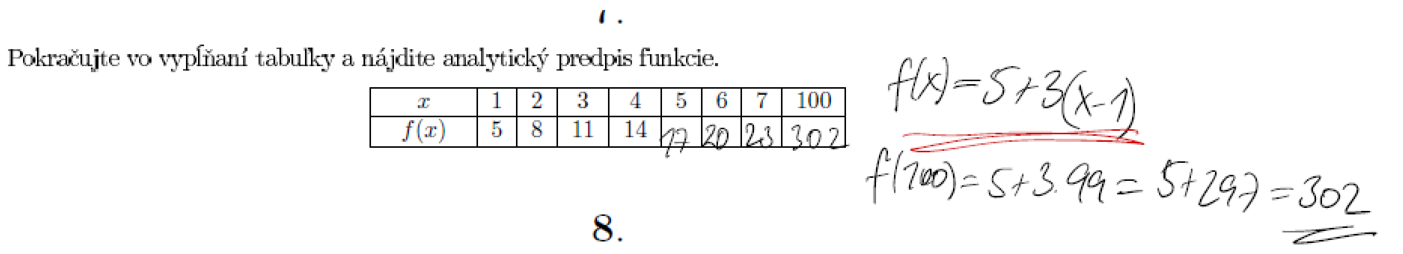
[illegible]

x	1	2	3	4	5	6	7	100
f(x)	5	8	11	14				

$\Rightarrow f(x) = 3x + 2$

Potom

x	1	2	3	4	5	6	7	100
f(x)	5	8	11	14	17	20	23	302



7.

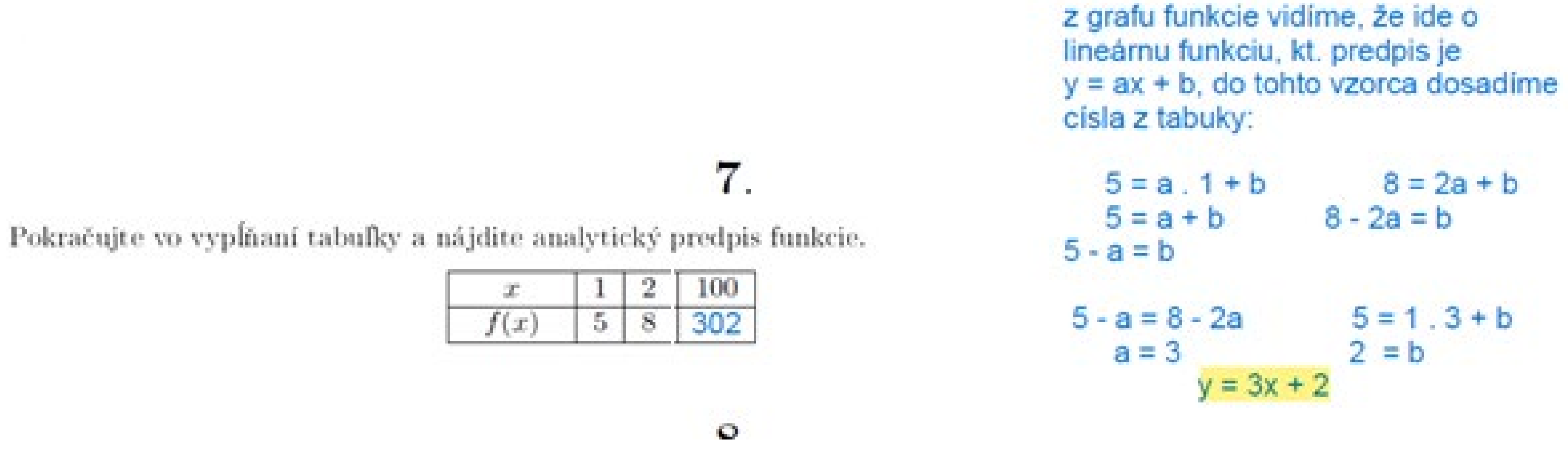
Pokračujte vo vypĺňaní tabulky a nájdite analytický predpis funkcie.

x	1	2	3	4	5	6	7	100
$f(x)$	5	8	11	14	17	20	23	302

$f(x) = 5 + 3(x-1)$

$f(100) = 5 + 3 \cdot 99 = 5 + 297 = 302$

8.



7.

Pokračujte vo vypĺňaní tabulky a nájdite analytický predpis funkcie.

x	1	2	100
$f(x)$	5	8	302

z grafu funkcie vidíme, že ide o lineárnu funkciu, kt. predpis je y = ax + b, do tohto vzorca dosadíme císla z tabuky:

5 = a . 1 + b
5 = a + b
5 - a = b

8 = 2a + b
8 - 2a = b

5 - a = 8 - 2a
a = 3

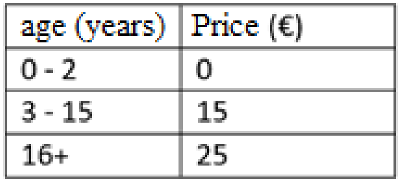
5 = 1 . 3 + b
2 = b

y = 3x + 2

Activity 8:
Verbal description of a function

Decide whether the relations described below are functions. Circle your answer and give reasons for your decision:

A. Dependence of theme park admission price on age as shown in the table:



age (years)	Price (€)
0 - 2	0
3 - 15	15
16+	25

Yes / No, Reason:

B. The area of the triangle ABC, which is assigned to the length of its side AB.

Yes / No, Reason:

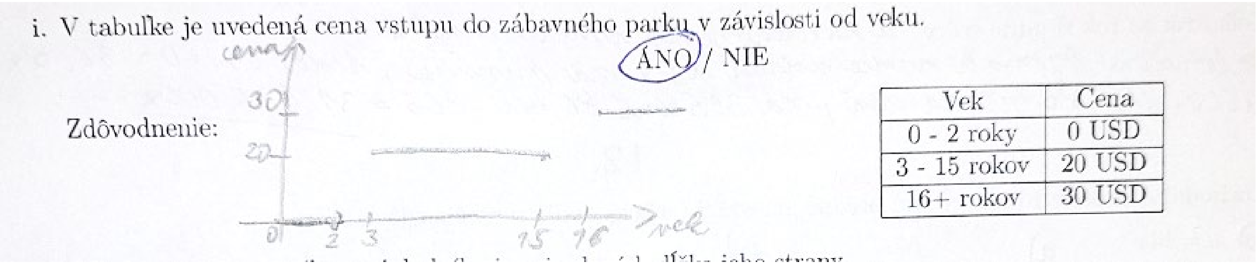
C. To the number of pages in a mathematics textbook is assigned the number of sentences on that page.

Yes / No, Reason:

D. To a student in your class are assigned the names of their siblings.

Yes / No, Reason:

Activity 8:
Verbal description of a function

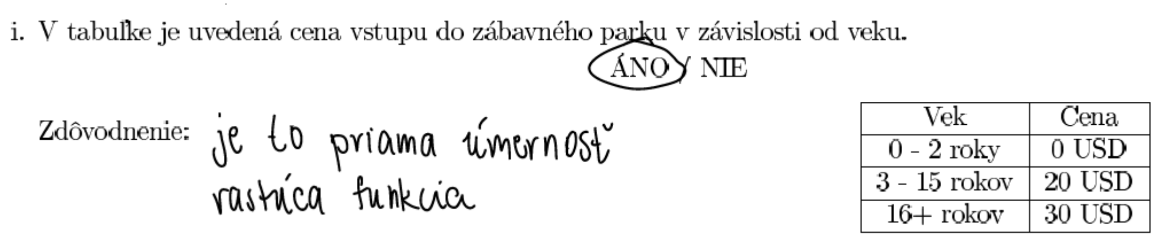


i. V tabulke je uvedená cena vstupu do zábavného parku v závislosti od veku.

[✓ ÁNO] / NIE

Zdôvodnenie:

Vek	Cena
0 - 2 roky	0 USD
3 - 15 rokov	20 USD
16+ rokov	30 USD

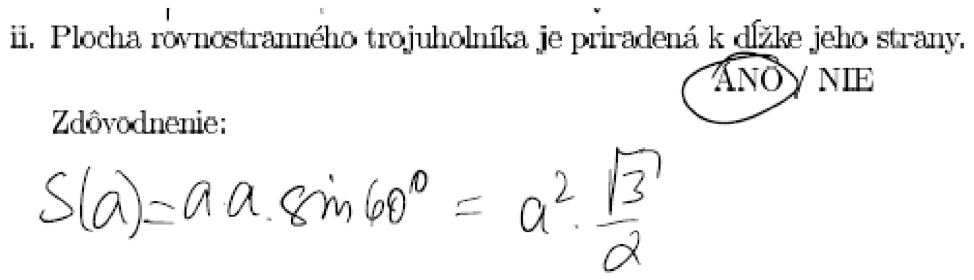


i. V tabulke je uvedená cena vstupu do zábavného parku v závislosti od veku.

[✓ ÁNO] / NIE

Zdôvodnenie: je to priama úmernosť
rastúca funkcia

Vek	Cena
0 - 2 roky	0 USD
3 - 15 rokov	20 USD
16+ rokov	30 USD

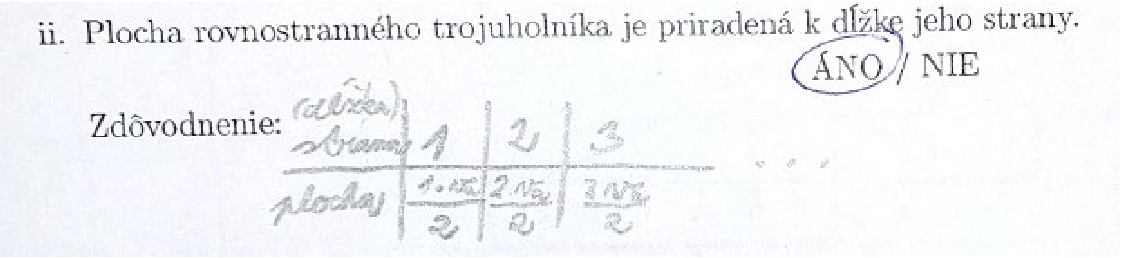


ii. Plocha rovnostranného trojuholníka je priradená k dĺžke jeho strany.

[✓ ÁNO] / NIE

Zdôvodnenie:

$$S(a)=a\,a\,\sin 60^\circ = a^2\cdot\frac{\sqrt{3}}{2}$$

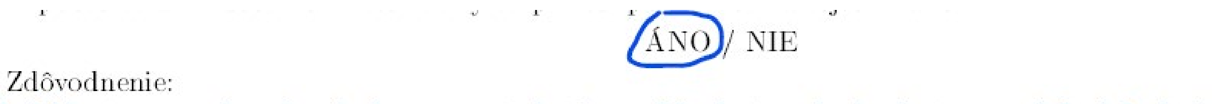


ii. Plocha rovnostranného trojuholníka je priradená k dĺžke jeho strany.

[✓ ÁNO] / NIE

Zdôvodnenie:

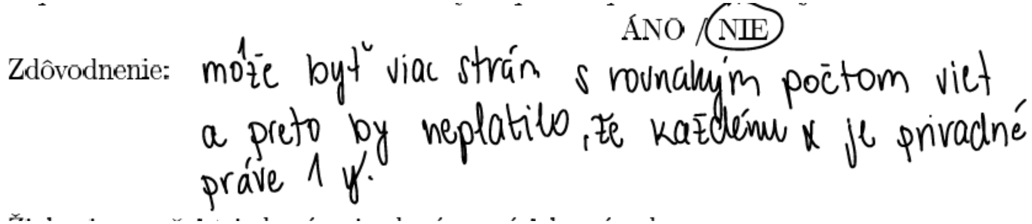
(dĺžka) strana	1	2	3
plocha	$\frac{1\cdot\sqrt{3}}{2}$	$\frac{2\cdot\sqrt{3}}{2}$	$\frac{3\sqrt{3}}{2}$



[✓ ÁNO] / NIE

Zdôvodnenie:

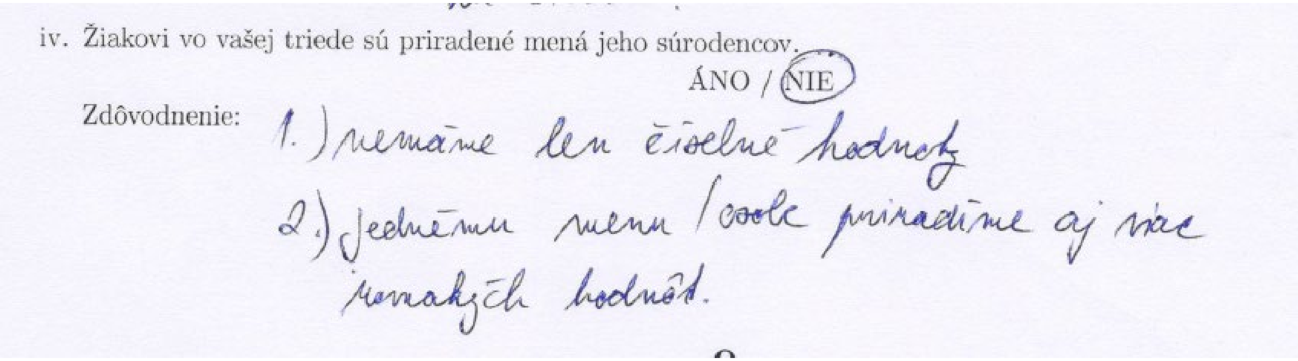
Každá strana má práve jeden pocet viet (nemôže byt na jednej strane aj 5 aj 6 viet)



ÁNO / [✓ NIE]

Zdôvodnenie: môže byť viac strán s rovnakým počtom viet a preto by neplatilo, že každému x je priradné práve 1 y.

Activity 8:
Verbal description of a function

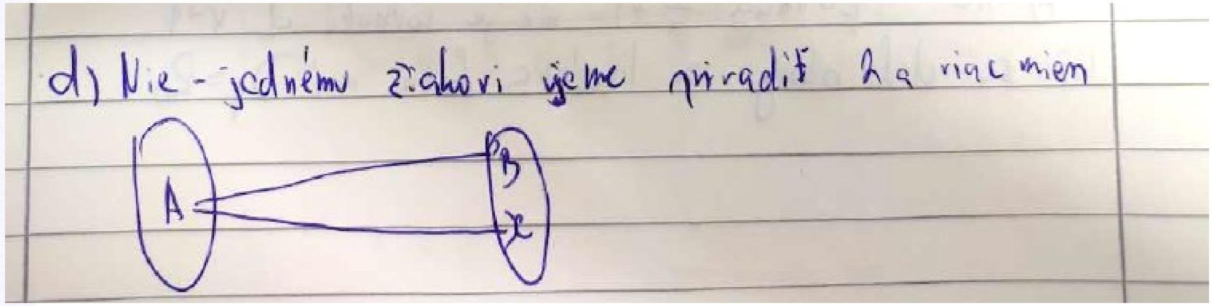


iv. Žiakovi vo vašej triede sú priradené mená jeho súrodencov.

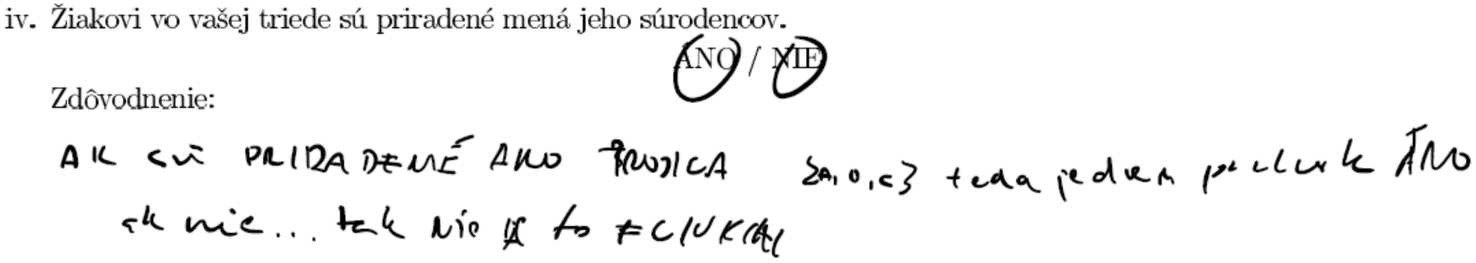
ÁNO / [✓ NIE]

Zdôvodnenie: 1.) nemáme len číselné hodnoty

2.) Jednému menu/osobe priradíme aj viac rovnakých hodnôt.



d) Nie - jednému žiakovi vieme priradiť 2 a viac mien



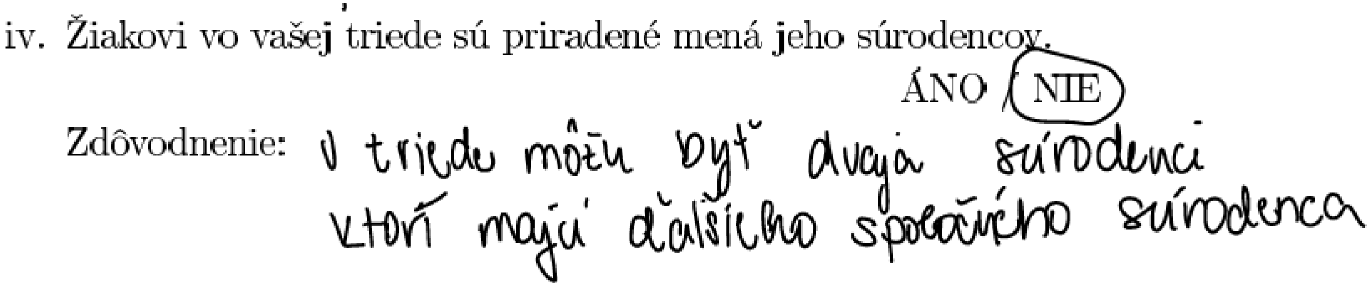
iv. Žiakovi vo vašej triede sú priradené mená jeho súrodencov.

[✓ ÁNO] / [✓ NIE]

Zdôvodnenie:

AK SÚ PRIRADENÉ AKO TROJICA {a, b, c} teda jeden prvok ÁNO

ak nie... tak NIE je to FUNKCIA



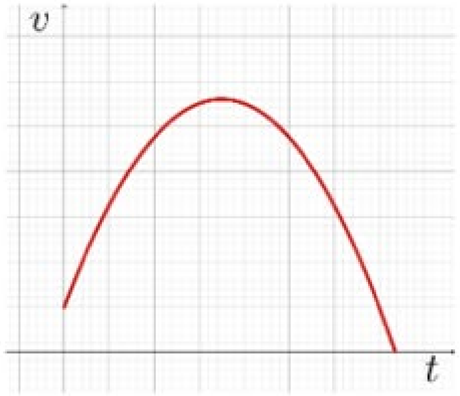
iv. Žiakovi vo vašej triede sú priradené mená jeho súrodencov.

ÁNO / [✓ NIE]

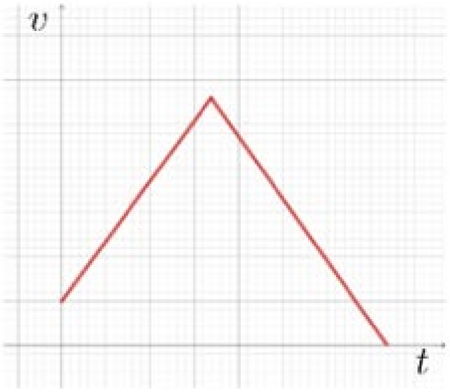
Zdôvodnenie: V triede môžu byť dvaja súrodenci ktorí majú ďalšieho spoločného súrodenca

Activity 9:
Stone

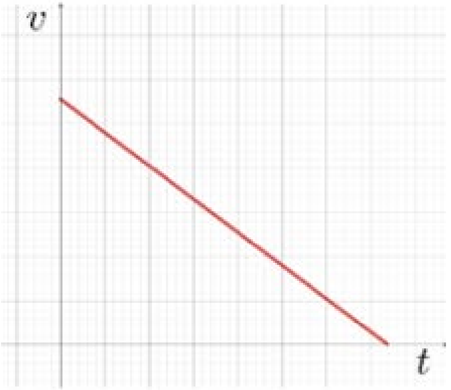
A stone was thrown vertically upwards. Select one of the graphs that illustrate the correspondence between velocity and time (neglecting air resistance). Justify your choice.



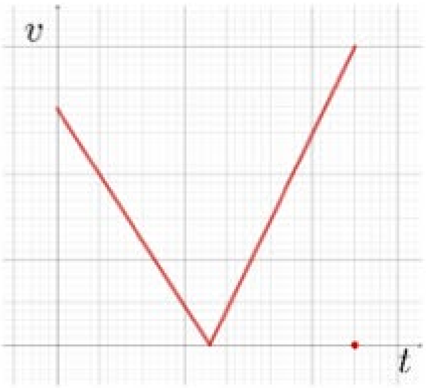
A



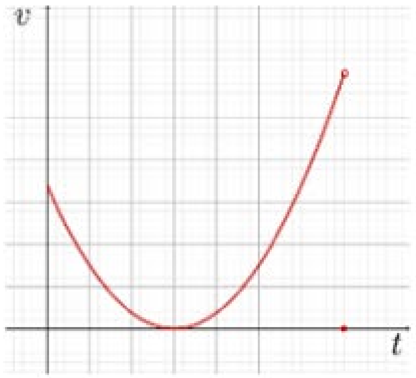
B



C

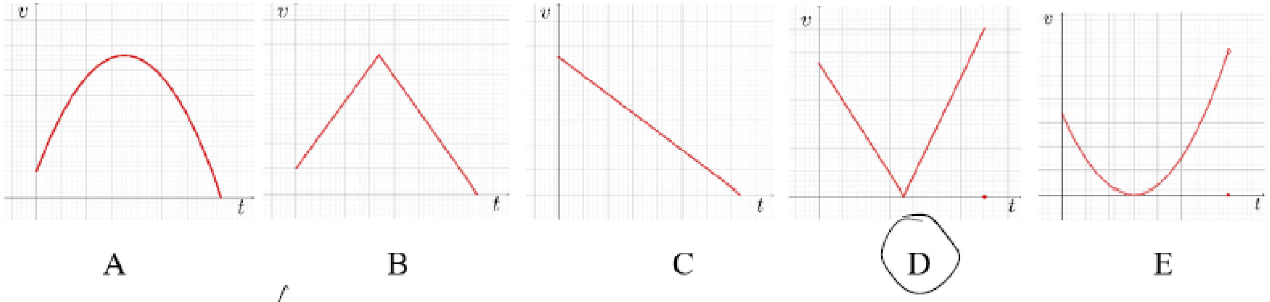


D



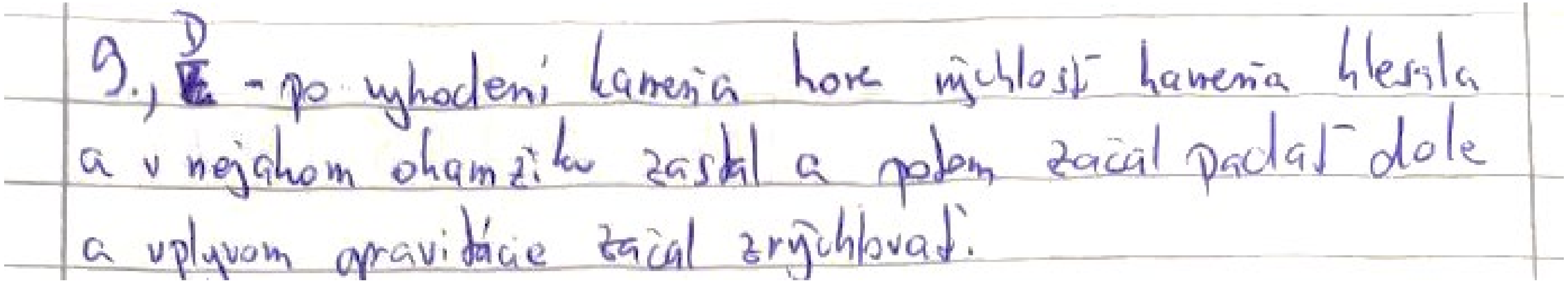
E

Activity 9:
Stone



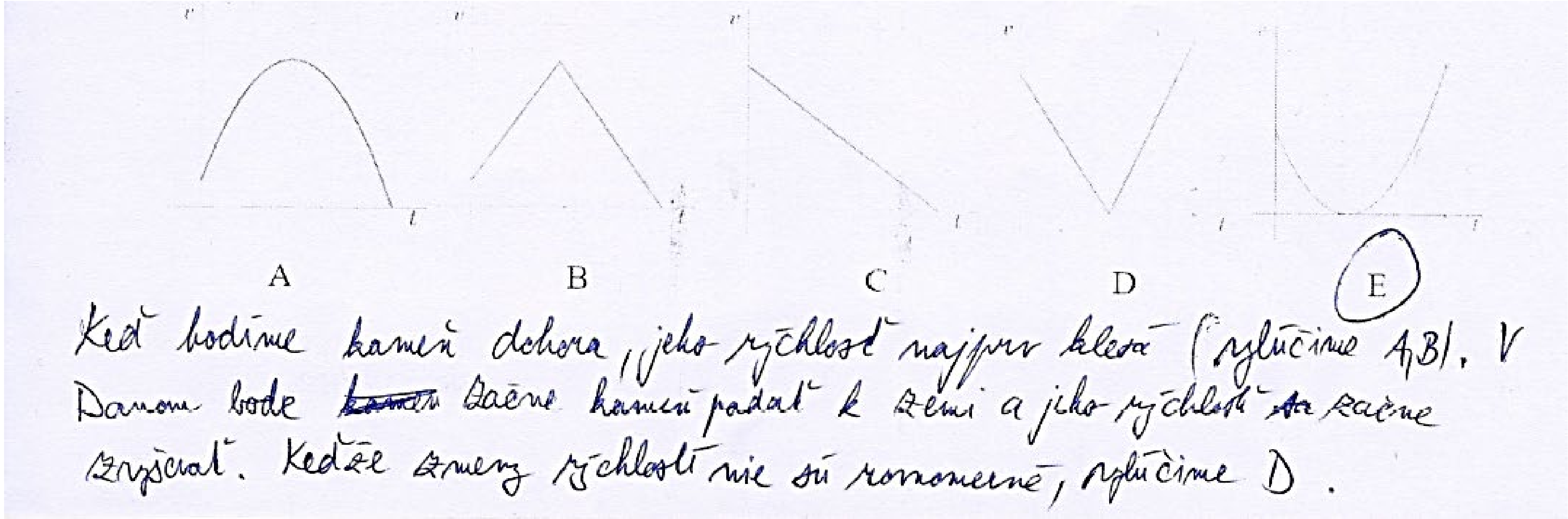
kameň vyhodíme rýchlosťou v_0, ako stúpa, rýchlosť mu klesá, v najvyššom bode zastane a padá naspäť dole so symetricky zväčšujúcou sa rýchlosťou

$\boxed{v = v_0 - gt}$ → lin. fcia

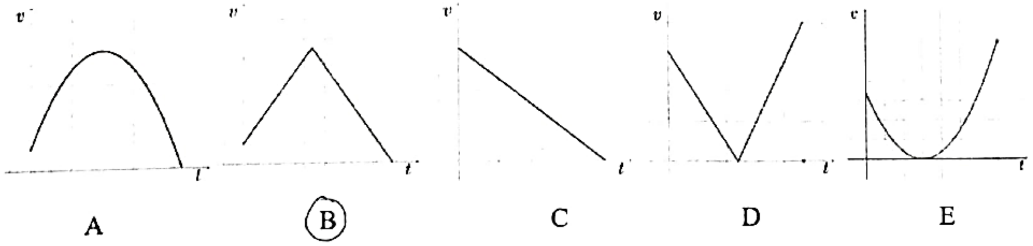


9, D ~~E~~ - po vyhodení kameňa hore rýchlosť kameňa klesala a v nejakom okamžiku zastal a potom začal padať dole a vplyvom gravitácie začal zrýchľovať.

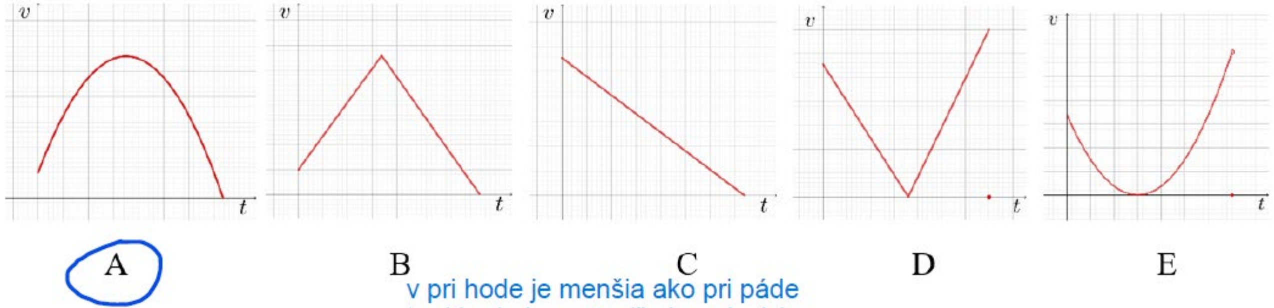
Activity 9:
Stone



Keď hodíme kameň dohora, jeho rýchlosť najprv klesá (vylúčime A,B). V Danom bode ~~kameň~~ začne kameň padať k zemi a jeho rýchlosť ~~sa~~ začne zvyšovať. Kedže zmeny rýchlosti nie sú rovnomerné, vylúčime D.



kameň ide hore a ďalej odo mňa, potom začne šikmo klesať



v pri hode je menšia ako pri páde
t pri hode je „väcsi" ako pri páde
kame padá rýchlejšie za kratší as

Activity 10:
Playground

Helenka wants to organize a birthday party in the children's playroom. She decides between the following offers:

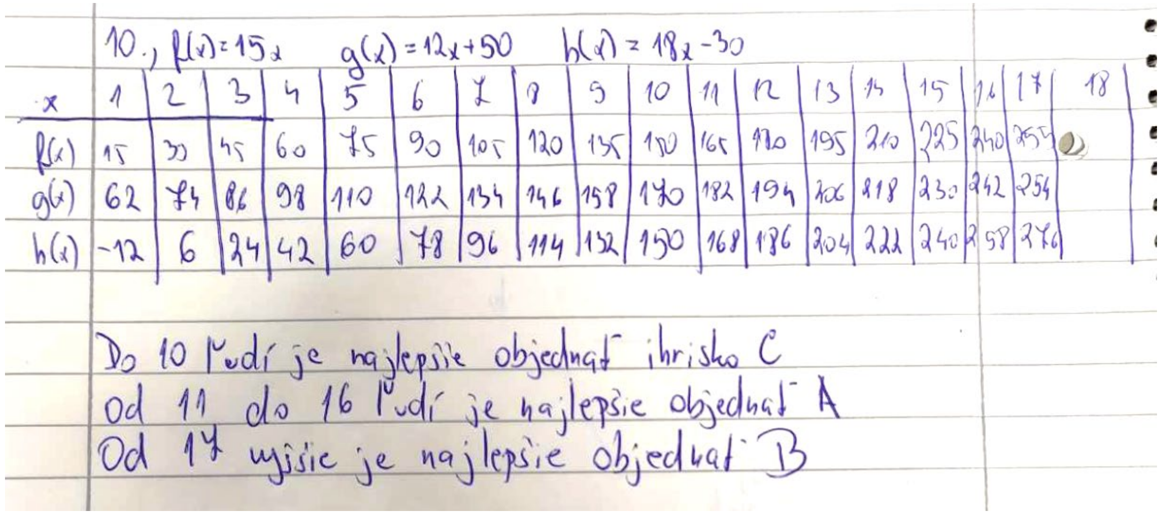
Playground A: The price for each guest is 15 €. 15 per guest.

Playground B: The price for each guest is 12 €. In addition, a fixed cost of 50 € is payable.

Playground C: The price for each guest is 18 €. A discount of 30 € will be given on the final price.

Which offer is the most advantageous for her?

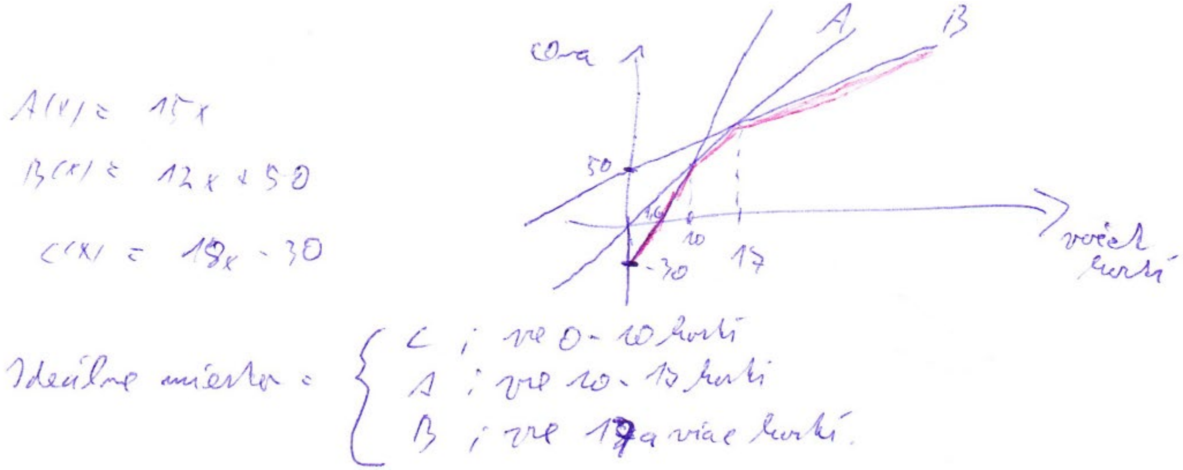
Activity 10:
Playground



10.) $f(x) = 15x$ $g(x) = 12x + 50$ $h(x) = 18x - 30$

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
f(x)	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	
g(x)	62	74	86	98	110	122	134	146	158	170	182	194	206	218	230	242	254	
h(x)	-12	6	24	42	60	78	96	114	132	150	168	186	204	222	240	258	276	

Do 10 ľudí je najlepšie objednať ihrisko C
Od 11 do 16 ľudí je najlepšie objednať A
Od 17 vyššie je najlepšie objednať B

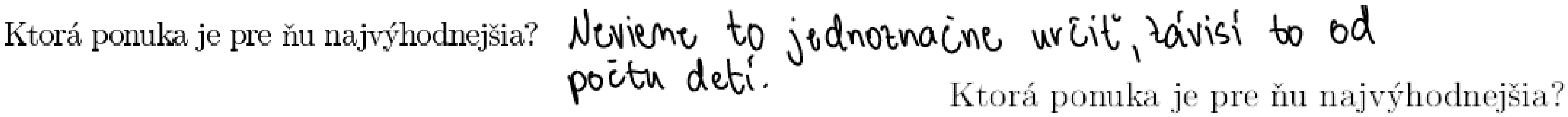


$A(x) = 15x$

$B(x) = 12x + 50$

$C(x) = 18x - 30$

Ideálne miesto: { C ; pre 0 - 10 hostí
A ; pre 10 - 17 hostí
B ; pre 17 a viac hostí.

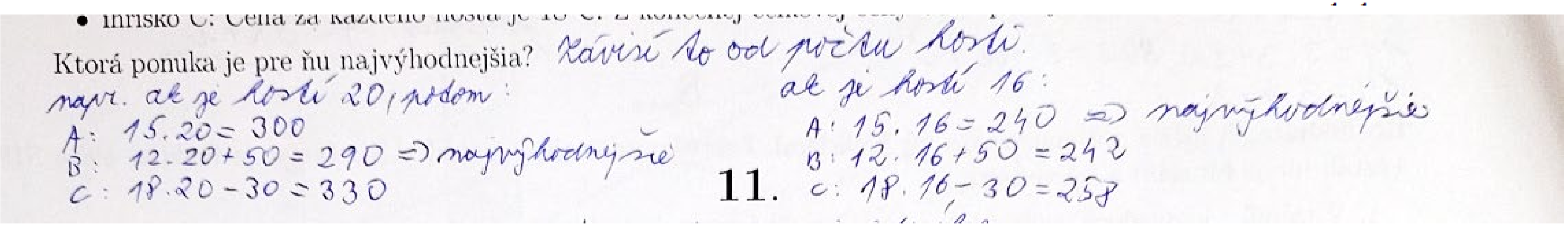


Ktorá ponuka je pre ňu najvýhodnejšia? Nevieme to jednoznačne určiť, závisí to od počtu detí.

Ktorá ponuka je pre ňu najvýhodnejšia?

a: y=15x , b: y=12x+50 , c: y=18x-30

závisí od poctu hostí



• Ihrisko C: Cena za každého hosťa je

Ktorá ponuka je pre ňu najvýhodnejšia? Závisí to od počtu hostí.

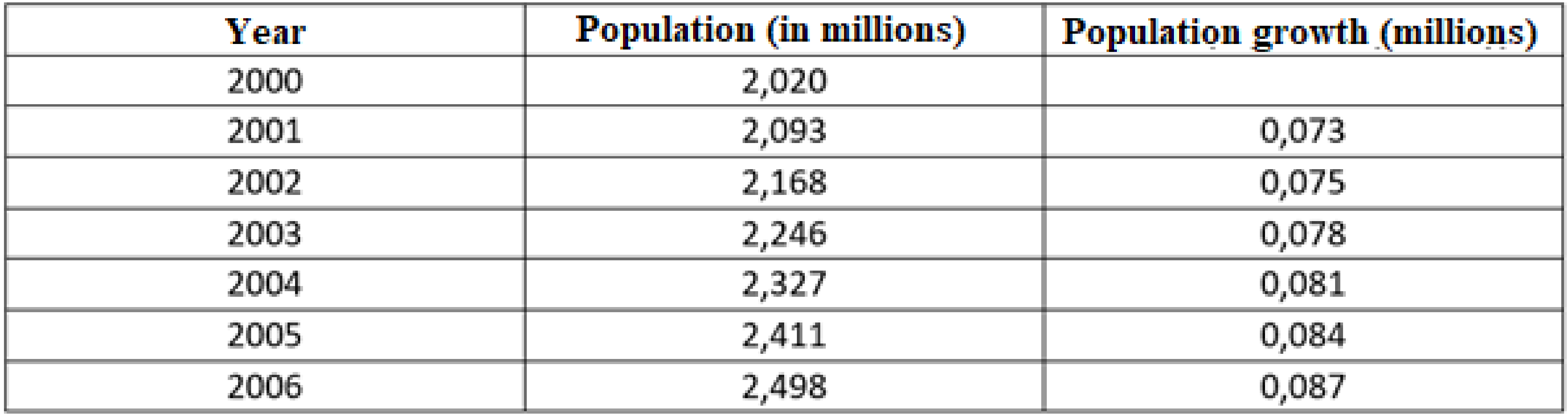
napr. ak je hostí 20, potom:
A: 15.20 = 300
B: 12.20 + 50 = 290 ⇒ najvýhodnejšie
C: 18.20 − 30 = 330

ak je hostí 16:
A: 15.16 = 240 ⇒ najvýhodnejšie
B: 12.16 + 50 = 242
C: 18.16 − 30 = 258

11.

Activity 11:
Population

The table shows Nevada's population from 2000 to 2006.



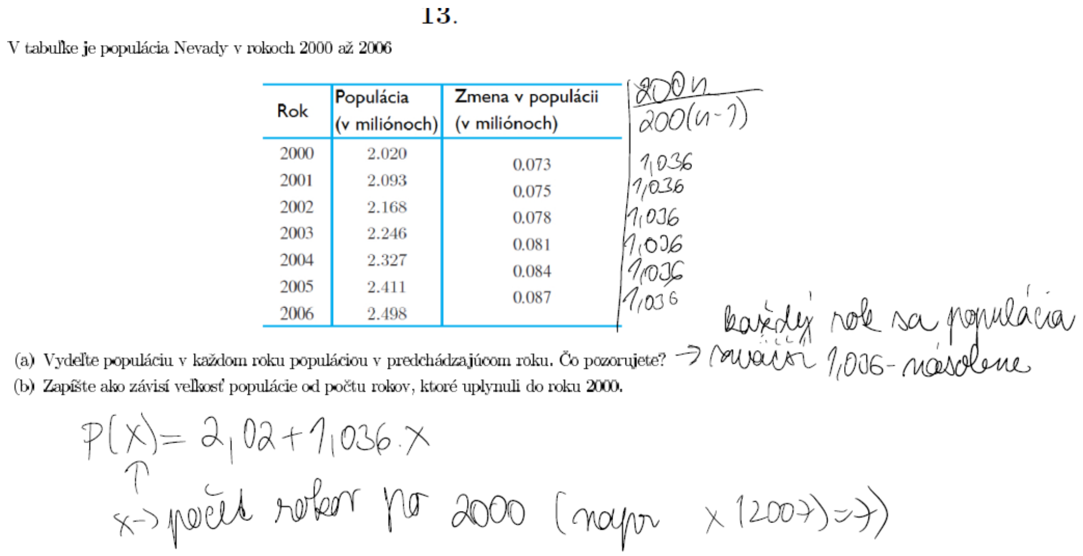
Year	**Population (in millions)**	**Population growth (millions)**
2000	2,020	
2001	2,093	0,073
2002	2,168	0,075
2003	2,246	0,078
2004	2,327	0,081
2005	2,411	0,084
2006	2,498	0,087

A. Divide the population in each year by the population in the previous year. What do you observe?

B. Write down how the size of the population depends on the number of years that have passed since 2000..

Activity 11:
Population

- - the ratio of two consecutive terms is about 1.035 - this satisfies the definition of a geometric sequence, or we can see that it is an exponential function.
- $p(x) = 2{,}02 \cdot 1{,}035^x$, where x is the number of years that have elapsed since 2000.



13.

V tabuľke je populácia Nevady v rokoch 2000 až 2006

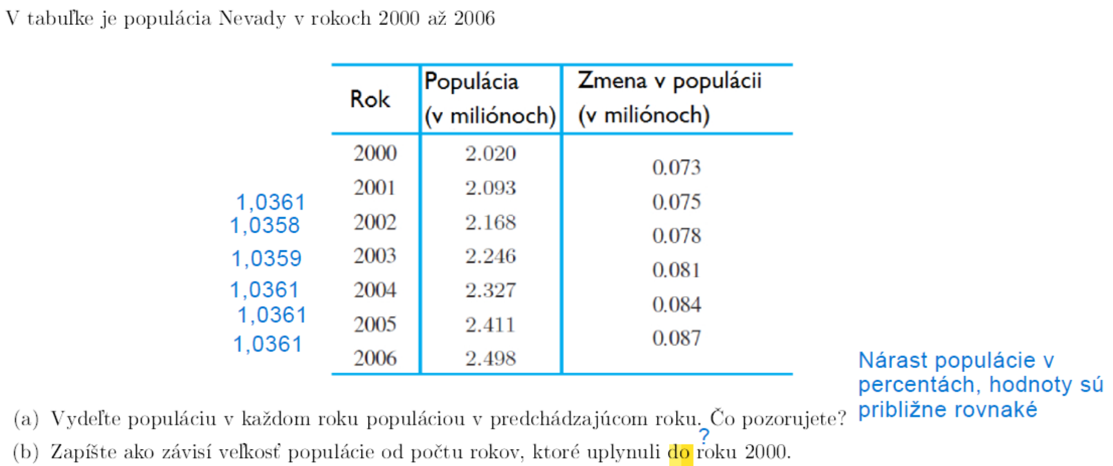
Rok	Populácia (v miliónoch)	Zmena v populácii (v miliónoch)	$\frac{200n}{200(n-1)}$
2000	2.020	0.073	1,036
2001	2.093	0.075	1,036
2002	2.168	0.078	1,036
2003	2.246	0.081	1,036
2004	2.327	0.084	1,036
2005	2.411	0.087	1,036
2006	2.498		

(a) Vydeľte populáciu v každom roku populáciou v predchádzajúcom roku. Čo pozorujete? → každý rok sa populácia zväčší 1,036 - násobne

(b) Zapíšte ako závisí veľkosť populácie od počtu rokov, ktoré uplynuli do roku 2000.

$P(X) = 2{,}02 + 1{,}036 \cdot X$

x -> počet rokov po 2000 (napr x(2007)=7)



V tabuľke je populácia Nevady v rokoch 2000 až 2006

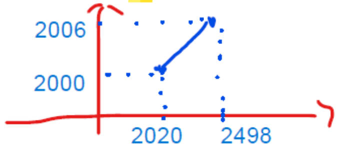
	Rok	Populácia (v miliónoch)	Zmena v populácii (v miliónoch)
	2000	2.020	0.073
1,0361	2001	2.093	0.075
1,0358	2002	2.168	0.078
1,0359	2003	2.246	0.081
1,0361	2004	2.327	0.084
1,0361	2005	2.411	0.087
1,0361	2006	2.498	

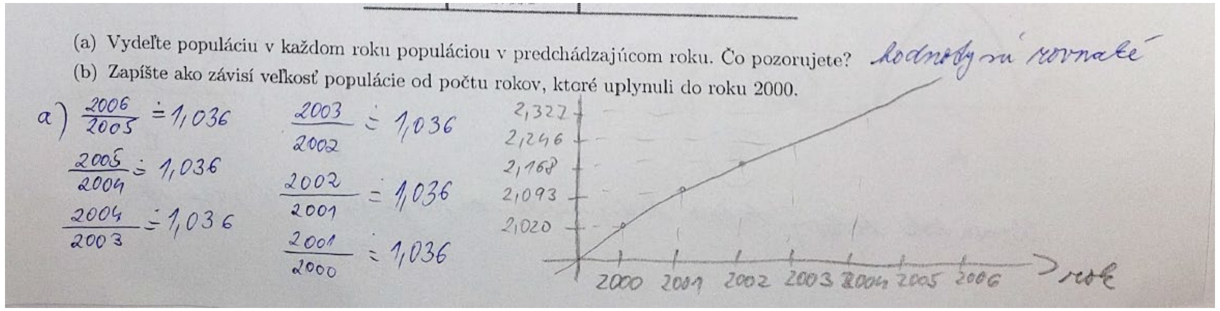
Nárast populácie v percentách, hodnoty sú približne rovnaké

(a) Vydeľte populáciu v každom roku populáciou v predchádzajúcom roku. Čo pozorujete?

(b) Zapíšte ako závisí veľkosť populácie od počtu rokov, ktoré uplynuli do ? roku 2000.

an=an-1*1,036





(a) Vydeľte populáciu v každom roku populáciou v predchádzajúcom roku. Čo pozorujete? hodnoty sú rovnaké

(b) Zapíšte ako závisí veľkosť populácie od počtu rokov, ktoré uplynuli do roku 2000.

a) $\frac{2006}{2005} \doteq 1{,}036$ $\frac{2003}{2002} \doteq 1{,}036$

$\frac{2005}{2004} \doteq 1{,}036$ $\frac{2002}{2001} \doteq 1{,}036$

$\frac{2004}{2003} \doteq 1{,}036$ $\frac{2001}{2000} \doteq 1{,}036$

Activity 12:
Slope

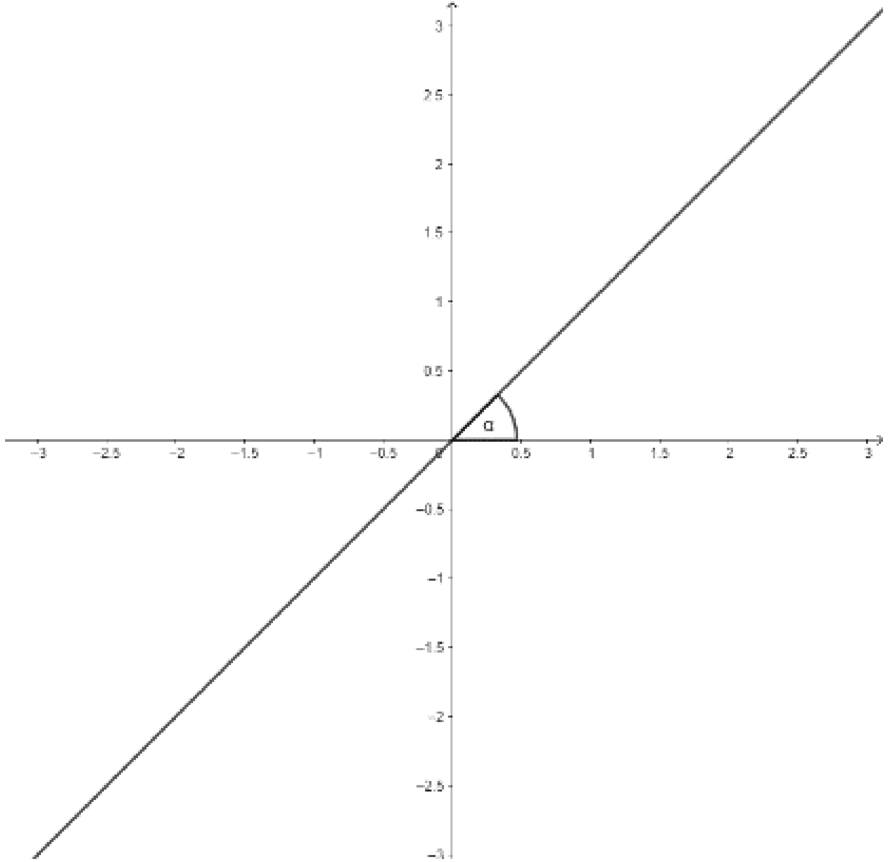
A. The graph represents the function f such that $f: x \longrightarrow x$.

I. What is the function slope? How did you determine it?

II. Divides the graph of a function f halves the angle bisected by the positive part of the coordinate axis x with the positive part of the coordinate axis y? Yes / No

Justification:

III. Can you calculate the tangent of the angle that the graph of a function f with the positive part of the coordinate axis x? If yes, determine its value. If not - because of what?



Activity 12:

Slope

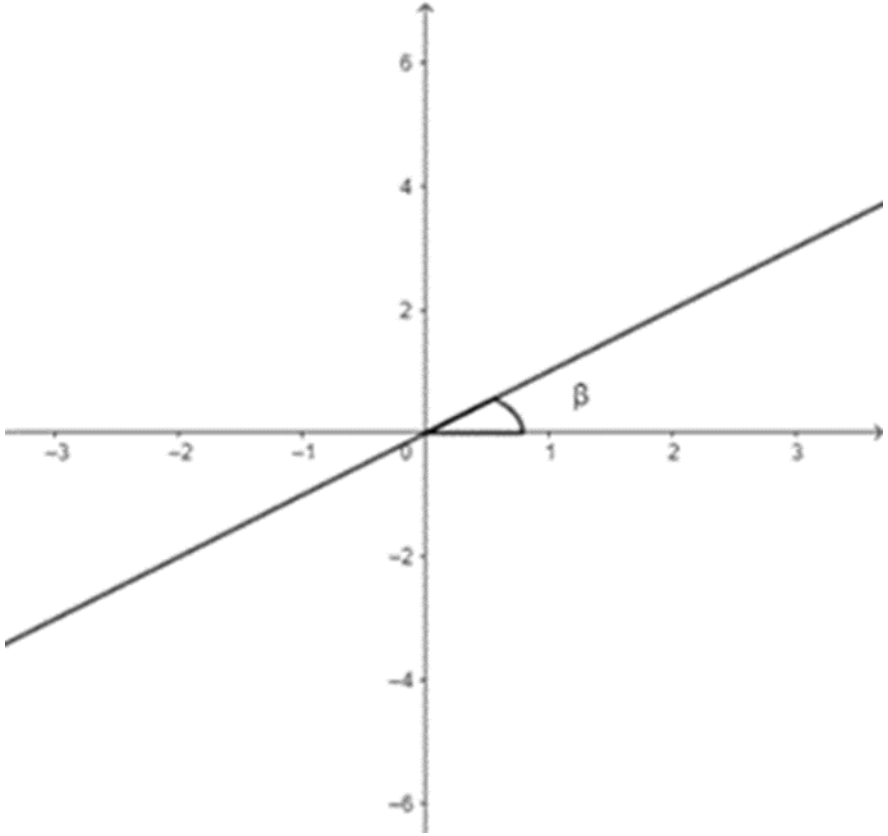
B. The student used software to draw the same functionf. It gave him the following graph:

I. What is the function slope? How did you determine it?

II. Divides the graph of a function f halves the angle bisected by the positive part of the coordinate axis x with the positive part of the coordinate axis y? Yes / No

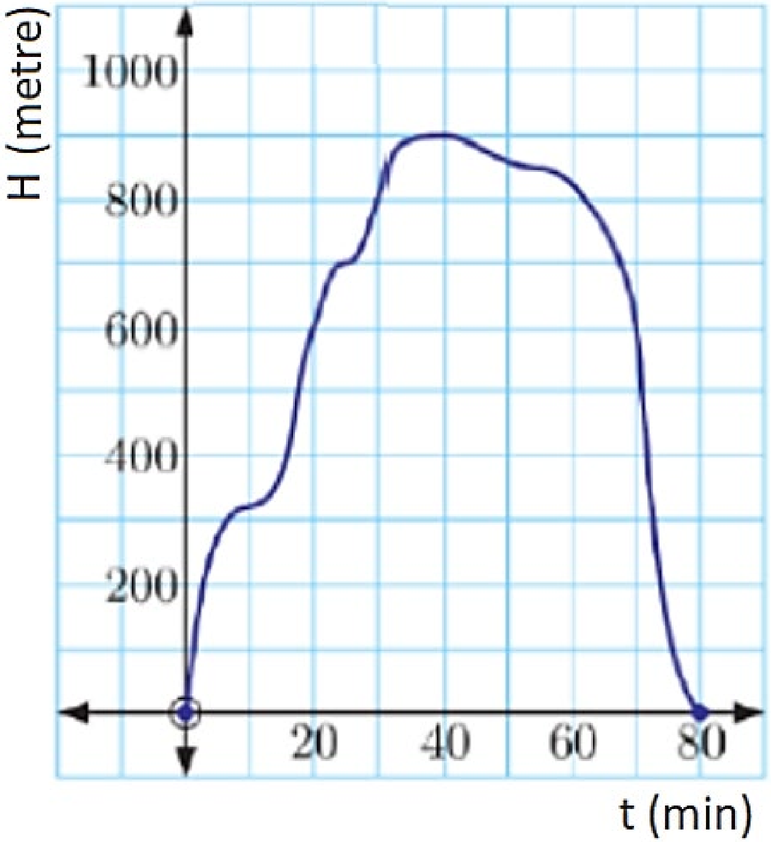
Justification:

III. Can you calculate the tangent of the angle that the graph of a function f with the positive part of the coordinate axis x? If yes, determine its value. If not - because of what?



Activity 13:
Balloon

When riding in a hot air balloon, the function indicates H the height of the balloon after t minutes of ride. Its graph is shown below:



a) Determine the value $H(30)$ and explain your answer in the context of a hot air balloon ride.

b) Find values of t such that $H(t) = 600$. Explain your answer in the context of a hot air balloon ride.

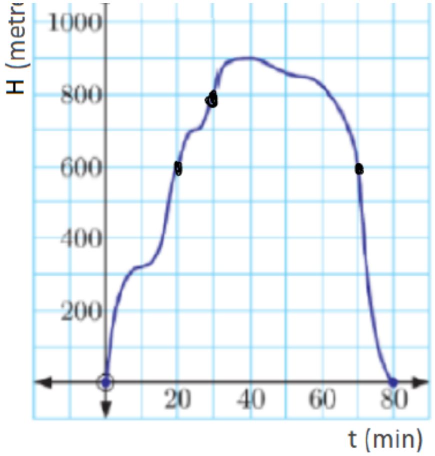
c) What range of heights was recorded for the balloon?

d) How long was the balloon ride?

e) Can you read the distance travelled by the balloon from the graph of function H ? Yes / No

Justification:

Activity 13:
Balloon



H(30)=800

teplovzdušný balón dosiahne za 30 min výšku 800m.

t = 20 a 70 v týchto časoch dosahuje výšku 600 m

(a) Určte hodnotu $H(30)$ a svoju odpoveď vysvetlite v kontexte letu teplovzdušným balónom.

(b) Nájdite také hodnoty t, aby $H(t) = 600$. Vysvetlite svoju odpoveď v kontexte letu teplovzdušným balónom.

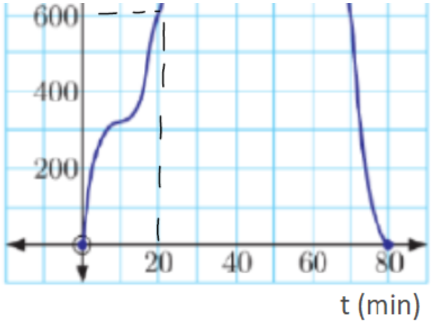
(c) Aký rozsah výšok bol zaznamenaný pre balón? → 0 - 900m

(d) Ako dlho trvala jazda balónom? → 80min

(e) Dokážete z grafu funkcie H odčítať vzdialenosť, ktorú balón prekonal? Zakrúžkujte ÁNO alebo NIE a zdôvodnite svoju odpoveď.

ÁNO / [✓ NIE]

Zdôvodnenie: Na základe údajov ktoré máme nevieme



(a) Určte hodnotu $H(30)$ a svoju odpoveď vysvetlite v kontexte letu teplovzdušným balónom. 700

↗ za 30 min sa balón dostal do výšky 700 m

(b) Nájdite také hodnoty t, aby $H(t) = 600$. Vysvetlite svoju odpoveď v kontexte letu teplovzdušným balónom. 20

za 20 min mal nadmorskú výšku 600 m

(c) Aký rozsah výšok bol zaznamenaný pre balón? 0-900

(d) Ako dlho trvala jazda balónom? 80 min

(e) Dokážete z grafu funkcie H odčítať vzdialenosť, ktorú balón prekonal? Zakrúžkujte ÁNO alebo NIE a zdôvodnite svoju odpoveď.

[✓ ÁNO] / NIE

Zdôvodnenie:

neviem ako ale dá sa

Activity 14:
Features of a function

A. Is there a function whose defining domain is (0,5) a the range of values is ⟨2,5⟩? If yes, draw its graph, write its prescription, or describe it in some other way.

B. Is there a function whose defining domain is the set of numbers {1,2,3} and the doma of values is the set {1,2}? If yes, draw its graph, write its prescription or describe it in some other way.

C. Is there a function whose defining domain is the set of numbers {1,2} and the doma of values is the set {1,2,3}? If yes, draw its graph, write its prescription, or describe it in some other way.

D. Is there a function that for any real numbers x satisfies the following requirements? If so, draw its graph or describe it in some other way.

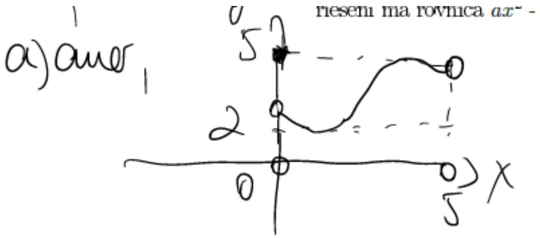
$$f(x+y) = f(x) \cdot f(y)$$

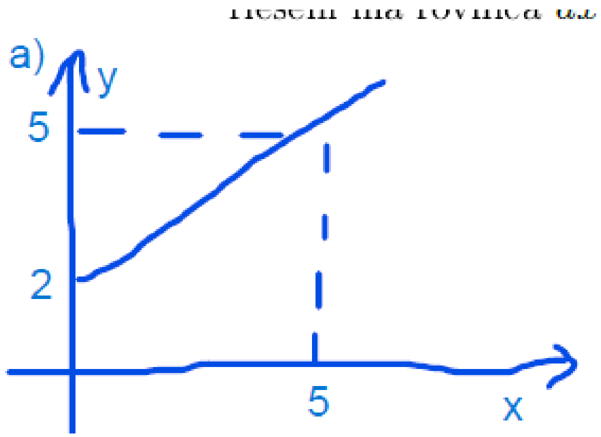
$$f(x+y) = f(x) + f(y)$$

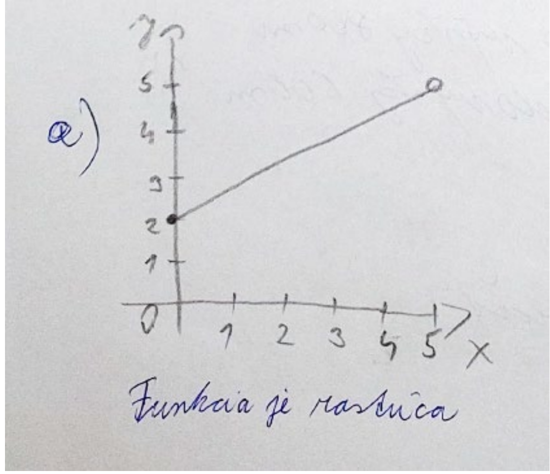
$$f(x \cdot y) = f(x) + f(y)$$

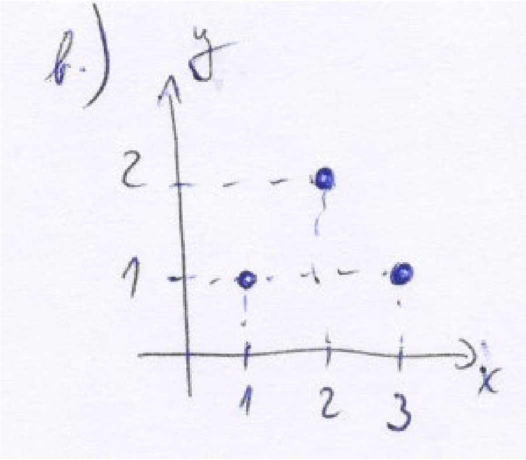
E. If we substitute 1 for x in the expression $ax^2 + bx + c$. we get a positive number, substituting 6 we get a negative number. How many solutions does the equation $ax^2 + bx + c = 0$ have? Justify your answer.

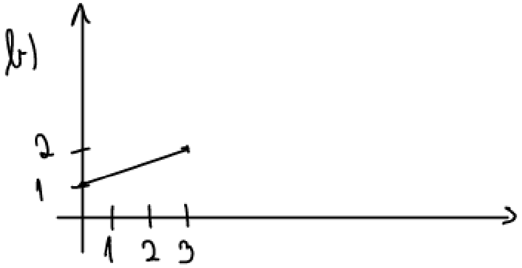
Activity 14:
Features of a function











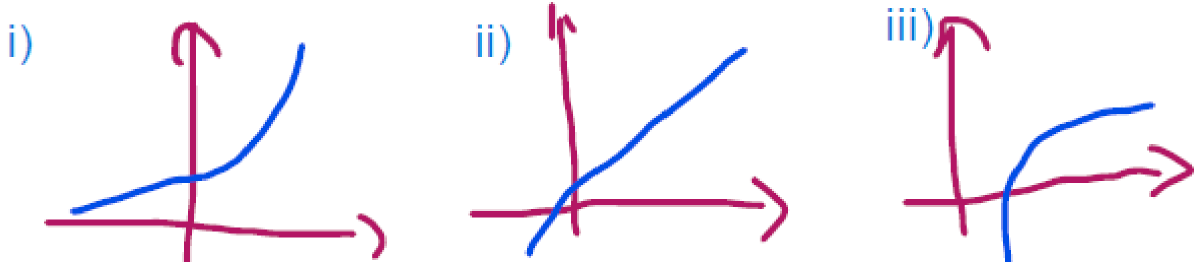
Activity 14:
Features of a function

(c) Existuje funkcia, ktorá pre akékoľvek reálne čísla x, y spĺňa nasledujúce požiadavky? (Ak áno, tak nakreslite jej graf, zapíšte jej predpis alebo ju popíšte inak.)

i. $f(x+y) = f(x) \cdot f(y)$ $y=e^x$

ii. $f(x+y) = f(x) + f(y)$ $y=x$

iii. $f(x \cdot y) = f(x) + f(y)$)? $y=\log x$



i) $e^{x+y} = e^x \cdot e^y \Rightarrow f(x)=e^x$

ii) $f(x) = 2x$

$2(x+y) = 2x+2y$

iii) $f(x) = \ln x$

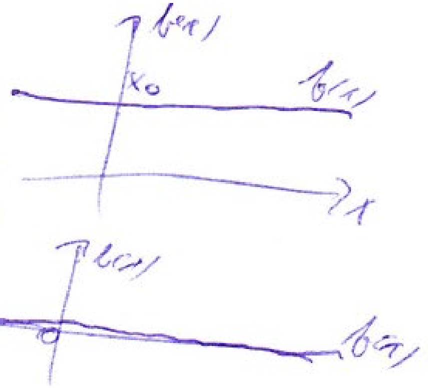
$\ln(x \cdot y) = \ln x + \ln y$

c) i) konštantná funkcia $f(x) = x_0$;

$x_0 \in \{1;0\}$

ii) konštantná nulová fcia $f(x) = 0$, $x \in R$

iii) analogicky ii)



Co-funded by the Erasmus+ Programme of the European Union

Functional Thinking: aspects, tasks, representations

Name of the course

Name of the Lector

Session number:

Intellectual output of FunThink (Erasmus+) project

Use one word to answer the question:
What is the goal of teaching and learning mathematics?

Use one word to answer the question:
What is the goal of teaching and learning mathematics?

THINKING

4 tasks

- Solve the following tasks in as many ways as possible
  - think about how younger or older pupils would solve the problem
- When solving the tasks, think about the following questions:
  - What are the differences between each solution?
  - How are the solutions for different tasks similar?
  - What knowledge and skills are needed to solve these tasks? Be as specific as possible.

4 tasks: Candle

We have the following situation:

“The candle is initially 24 cm tall. It shrinks by 2 cm every hour.” How can you mathematically express the situation?

adapted, original task by R. Nitsch, www.codi-test.de

Activity:

Solve this task in as many ways as you can. Write down your solutions.

4 tasks: Cards

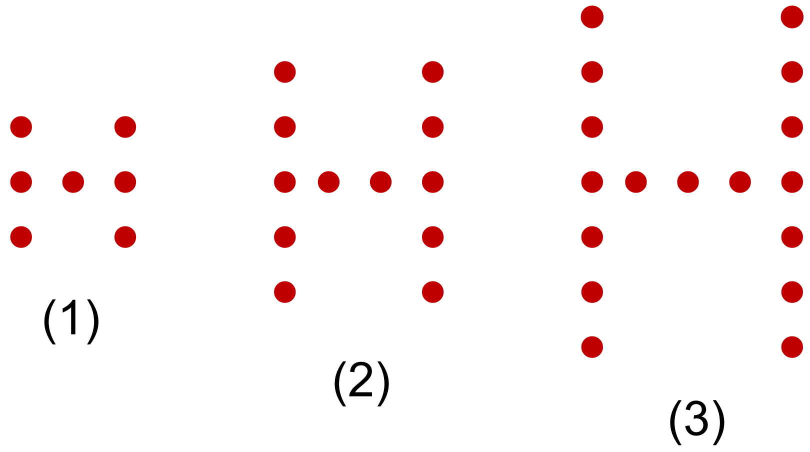
Emil has set the displayed first three patterns with circle cards.

a) What will be the amount of cards in the fifth pattern?

b) What formula can be used in order to determine the amount of cards for any of the patterns? Tick the correct box.

☐ $s = n + 6$
☐ $s = 5n + 2$
☐ $s = 3n + 4$

n indicates the pattern number
s indicates the number of chips



Activity:

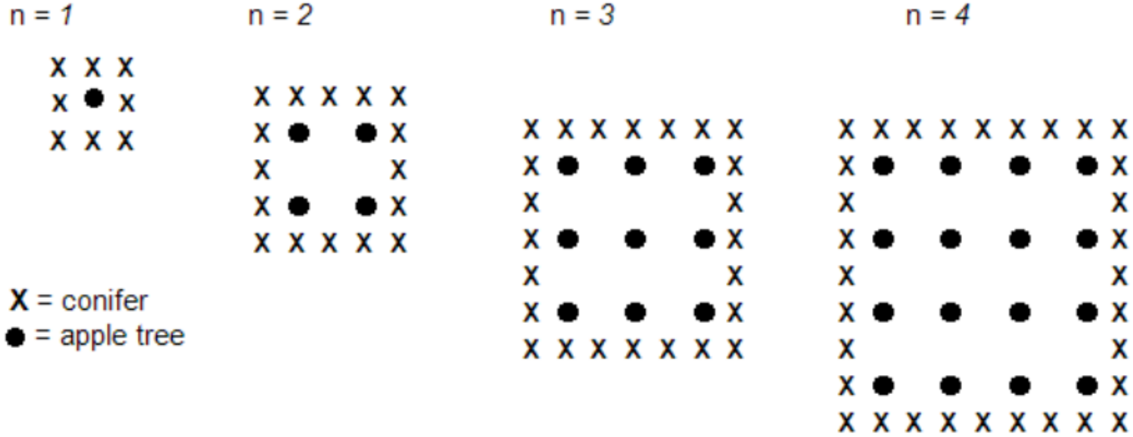
Solve this task in as many ways as you can. Write down your solutions.

Adapted from a task of the Realschule final examination, Germany

4 tasks: Orchard

A farmer plants apple trees in a square pattern. In order to protect the trees against the wind he plants conifers all around the orchard (see picture).
Complete the table:

n	Number of apple trees	Number of conifers
1	1	8
2	4	
..	…	…



When (for what *n*) will be the number of conifers be the same as the number of apple trees?

Activity:

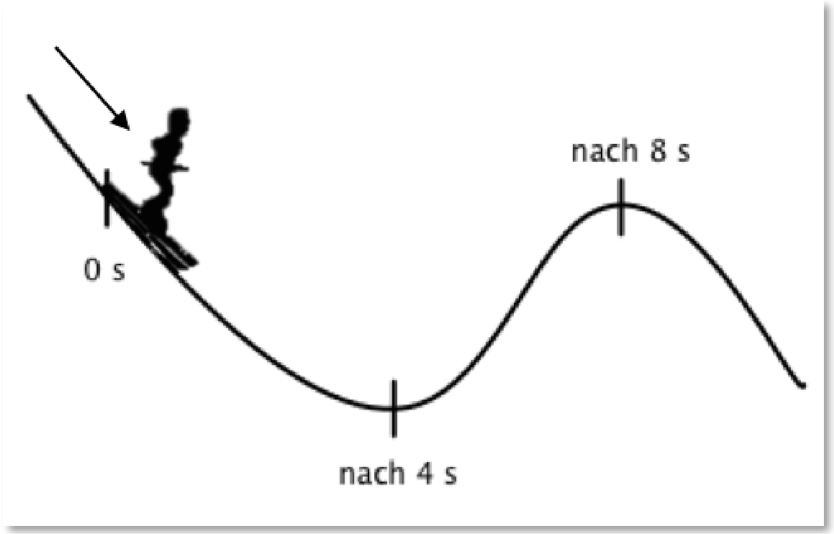
Solve this task in as many ways as you can. Write down your solutions.

OECD, 2009, p. 102

Four tasks: Skier

Activity:

Solve this task in as many ways as you can. Write down your solutions..



(cf. Barzel & Ruchniewicz, 2020, p. 9)

A skier already has an initial speed at the time 0 seconds and then let himself glide down the hill (without intentionally braking).

A. When is he faster: at second 4 or 8?
B. Describe in words how his speed changes with time.
C. A second skier is traveling the same distance and has a higher initial speed than the first skier has. Compare the two runs in words.

Aspects of functional thinking and representations

Discussion:

What are the differences between different solutions of the same task~~s~~?

What makes solutions of different tasks similar?

Representations used.

The way of looking at functions.

Aspects of functional thinking and representations

Discussion:

What knowledge and skills are needed to solve these tasks?

Pupils must describe and interpret functional relationships using different **representations**.
Consequently: link and change these representations

Pupils need to consider functional relationships from different **perspectives/aspects**.

Aspects of function and representation

Pupils must describe and interpret functional relationships in different **representations**.
Consequently: link and change these representations

Discussion:

What knowledge and skills are needed to solve these tasks?

The importance of representations:
Provide access to a mathematical object that is needed for concept formation and problem solving (e.g. Pittalis et al., 2020; Hußmann & Laakman, 2011)

- Understanding
- Link
- Change

... representations of functions / functional relationships

Typically used function representations:

- Graphs
- Equations
- Tables
- Descriptions and pictures of situations ...

Aspects of function and representation

Discussion:

What knowledge and skills are needed to solve these tasks?

Pupils need to think about functional relationships from different **perspectives/aspects**.

Aspects: Malle, 2000; Pittalis et al., 2020; Vollrath, 1989)

- ***Input-output:*** exploring how a certain input value will lead to an output value; finding a rule
- ***Correspondence***: understanding the relationship between independent and dependent variables and being able to represent it; more formal definition of a function as a set of ordered pairs.
- ***Covariation***: the study of how one variable changes when another one varies.
- ***Object***: a function is considered as a mathematical object with its own representations and properties that can be compared to other mathematical objects or functions.

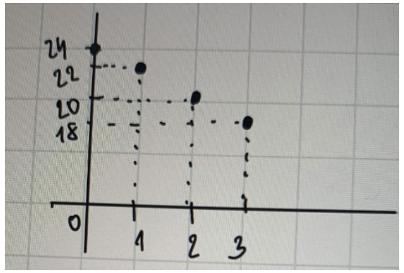
Aspects of function and representation: input - output

Aspects: Malle, 2000; Pittalis et al., 2020; Vollrath, 1989)

- ***Input-output:*** exploring how a certain input value will lead to an output value; finding a rule.

Representations:

- Chain of numbers 24 -2 22 -2 20 -2 18 -2 ... ...
- Graphical: Point in the coordinate system



- Table without perceptions of any other relationships
- Function machine
- ~~General formula~~

We have the following situation:

"The candle is initially 24 cm high. Every hour it shrinks by 2 cm."

How can you mathematically express the situation?

Aspects of function and representation: correspondence

Aspects: Malle, 2000; Pittalis et al., 2020; Vollrath, 1989)

- ***Correspondence***: understanding the relationship between independent and dependent variables and being able to represent it; a more formal definition of a function as a set of ordered pairs.

We have the following situation:

"The candle is initially 24 cm high. Every hour it shrinks by 2 cm."

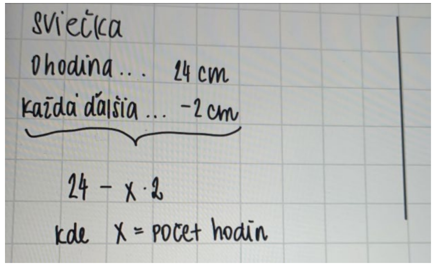
How can you mathematically express the situation?

Representations:

- Table - if the pupil perceives individual ordered pairs

Time	0	1	2
Height of the candle	24	22	20

- General regulation

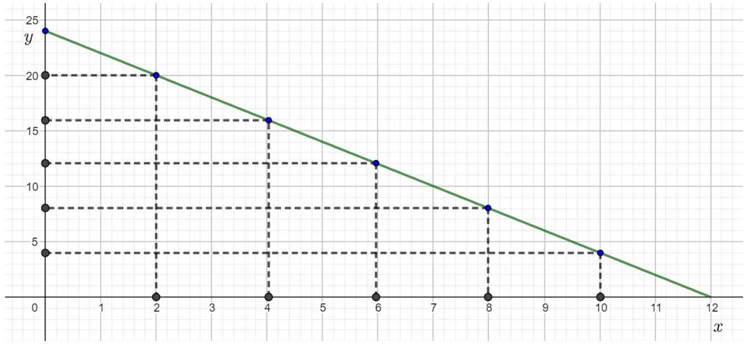


Sviečka
0 hodina ... 24 cm
každá ďalšia ... -2 cm

$$\underbrace{}$$

24 - x · 2
kde x = pocet hodín

- Graph
  Graph a function where the pupil perceives ordered pairs *x* and f(*x*)



Aspects of function and representation: correspondence

Aspects: Malle, 2000; Pittalis et al., 2020; Vollrath, 1989)

- ***Correspondence***: understanding the relationship between independent and dependent variables and being able to represent it; a more formal definition of a function as a set of ordered pairs.

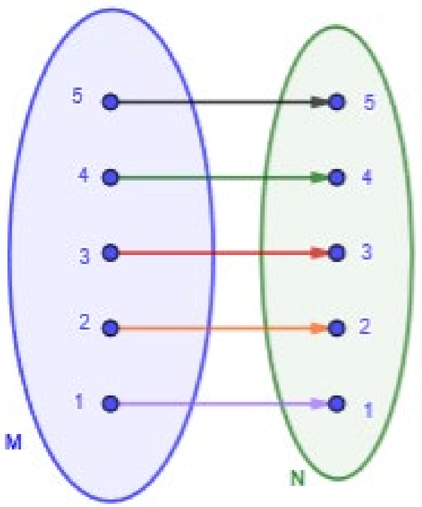
We have the following situation:

"The candle is initially 24 cm high. Every hour it shrinks by 2 cm."

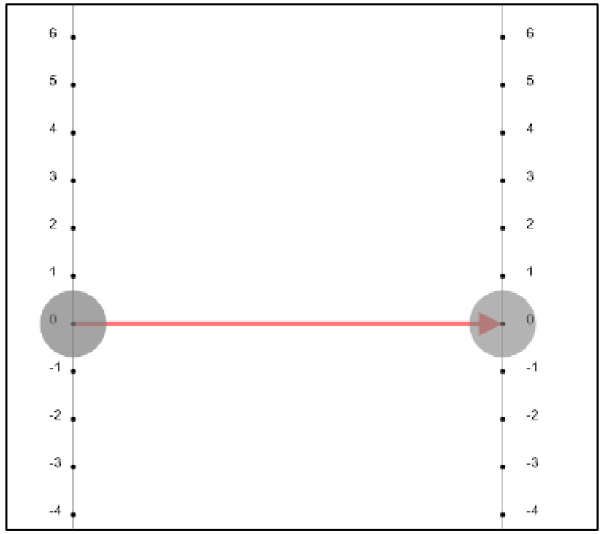
How can you mathematically express the situation?

Representations:

- Venn diagram with arrows



- Nomogram



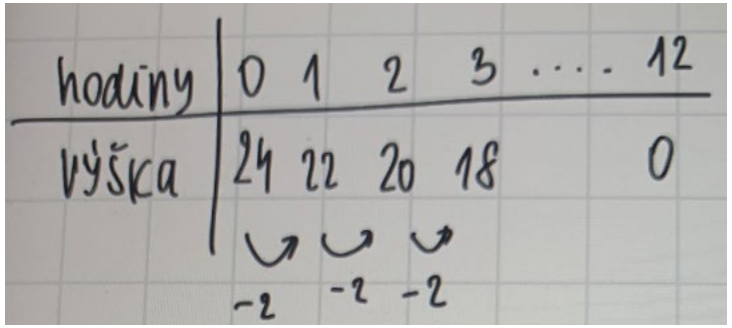
Aspects of function and representation: covariation

Aspects: Malle, 2000; Pittalis et al., 2020; Vollrath, 1989)

- ***Covariation***: the study of how one variable changes when another changes.

Representations:

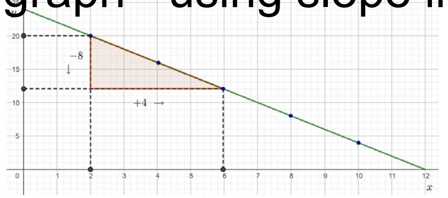
- Table - if the pupil perceives changes in individual variables



hodiny	0	1	2	3	....	12
výška	24	22	20	18		0

-2 -2 -2

- Graphically - reading the change and rate of change from the graph - using slope in argumentation



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How can you mathematically express the situation?

Aspects of function and representation: object

Aspects: Malle, 2000; Pittalis et al., 2020; Vollrath, 1989)

- ***Object***: a function is considered a mathematical object with its own representations and properties that can be compared with other mathematical objects or functions.

We have the following situation:

"The candle is initially 24 cm high. Every hour it shrinks by 2 cm."

What formula would you use to express this situation?

Representations:

- Formula - use of general formulas of functions (e.g. $y = ax + b$ is linear function)
  - Identification of the properties of a given function (decreasing - negative *„a"*)
- Graphically - using the properties of a given class of functions (the graph of a linear function is a straight line)
  - Identifying the properties of a given function (decreasing - correct slope of the line)
  - comparing different graphs

Task

- Draw the graph of the following function: $y = 2x + 3, x \in \mathbb{R}$.

Activity:

Solve this problem. Produce more solutions, make sure you use different aspects of the functional thinking.

Also write down the thought process you have in mind.

Four tasks: Cards

Emil has set the displayed first three patterns with circle cards.

a) What will be the amount of cards in the fifth pattern?

b) What formula can be used in order to determine the amount of cards for any of the patterns? Tick the correct box.

☐ $s = n + 6$
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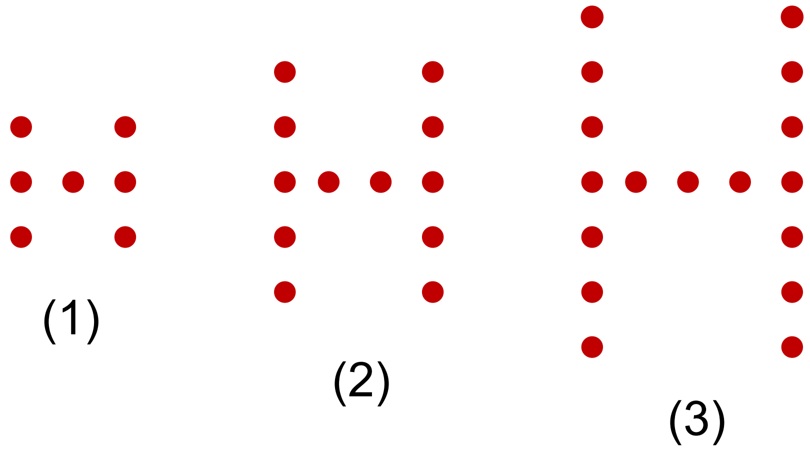

Adapted from a task of the Realschule final examination, Germany

4 tasks: Orchard

A farmer plants apple trees in a square pattern. In order to protect the trees against the wind he plants conifers all around the orchard (see picture).
Complete the table:

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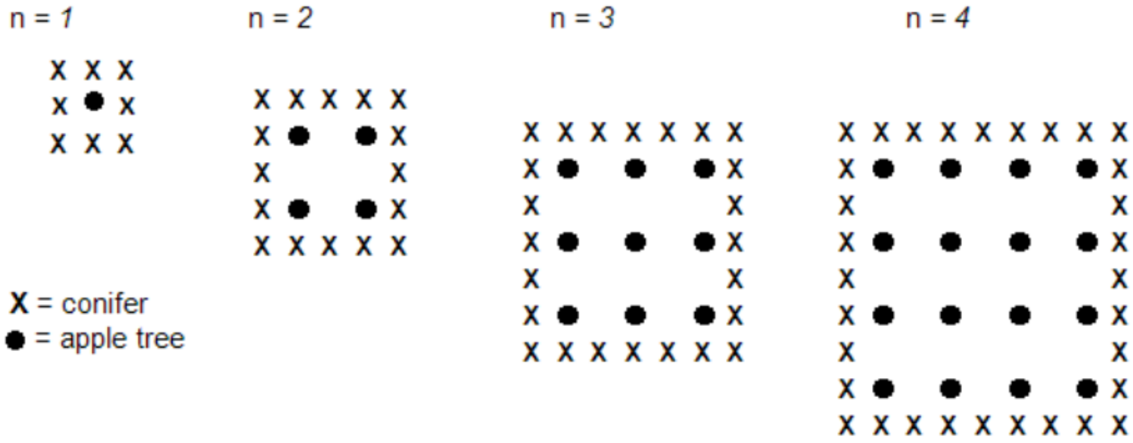
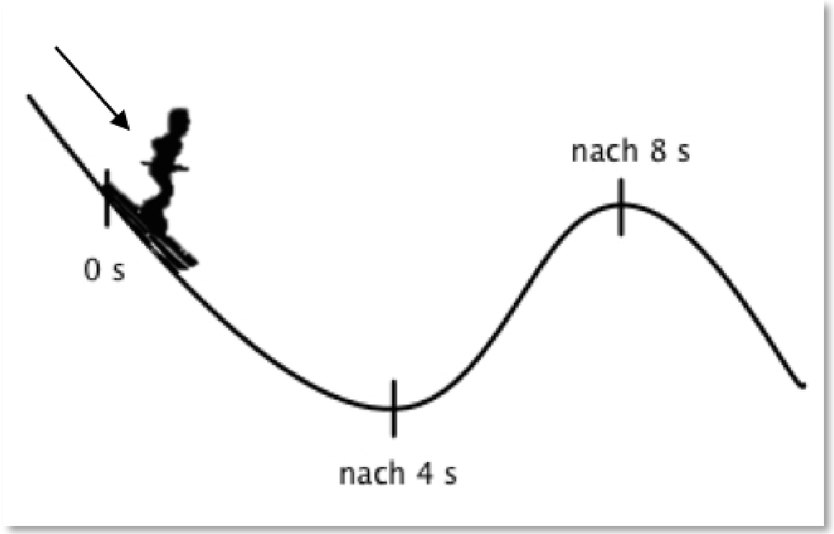
2	4	
..	…	…



When (for what *n*) will be the number of conifers same as the number of apple trees?

OECD, 2009, p. 102

4 tasks: Skier



(cf. Barzel & Ruchniewicz, 2020, p. 9)

A skier already has an initial speed at the time 0 seconds and then let himself glide down the hill (without intentionally braking).

A. When is he faster: at second 4 or 8?
B. Describe in words how his speed changes with time.
C. A second skier is traveling the same distance and has a higher initial speed than the first skier has. Compare the two runs in words.

Questions for reflection

- Which tasks would you choose if you wanted **to see which aspects students naturally prefer?**
  A. Candle
  B. Buttons/Card
  C. Orchard
  D. Skier

- Which tasks would you choose if you wanted **to find out if pupils can use the covariation aspect to solve a task?**
  A. Candle
  B. Buttons/Card
  C. Orchard
  D. Skier

Back to the “set of tasks”

Which tasks would you choose if you wanted to see:

- **input-output aspect in the solution?**
- **covariation aspect in the solution?**
- **correspondence aspect in the solution?**
- **object aspect in the solution?**
- **which aspect students prefer?**

Poster about functional thinking

- **In a group, create a digital poster about one of the aspects of functional thinking.**
- **Include all relevant information: representations, tasks, solutions, etc.**

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Functional Thinking in Curriculum

Name of the course

Name of the Lector

Session number:

Intellectual output of FunThink (Erasmus+) project

What does “Development of functional thinking” mean?

What does “Development of functional thinking” mean?

ASPECTS

- to perceive the concept of function in its diversity including the aspects of functional thinking:
  - **input-output**
  - **covariance**
  - **correspondence**
  - **object**
- using the appropriate aspect for the given situation

REPRESENTATIONS

- understanding different forms of representations
  graph
  formula
  table
  verbal description
  ...
  change of representation
  choosing the appropriate representation for the situation

APPLICATIONS

- see and use functions and functional thinking within mathematics
- see and use functions and functional thinking outside mathematics

Task - Curricular material study

- Each group is assigned a type of school for the following tasks.

A. Study the curriculum documents for the given type of school.

B. Distribute the 12 points between the aspects of functional thinking according to their importance at the given type of school / grade level.

C. For each representation, indicate whether it is used at the given type of school/ grade level (P – used, I – used "introductory", "preparatory", "propaedeutic", X – not used).
In the free space, explain important information. For example, what kind of propaedeutic is used, why is the representation not used, etc. Moreover, state what other representations of functions occur.

D. List specific mathematical topics in which functional thinking can be developed. Explain how this can be accomplished.
  - within the scope of the function
  - outside of the scope of function

E. List several practical applications of functional thinking appropriate for the given type of school/ grade level. Focus also on the curriculum of other school subjects.

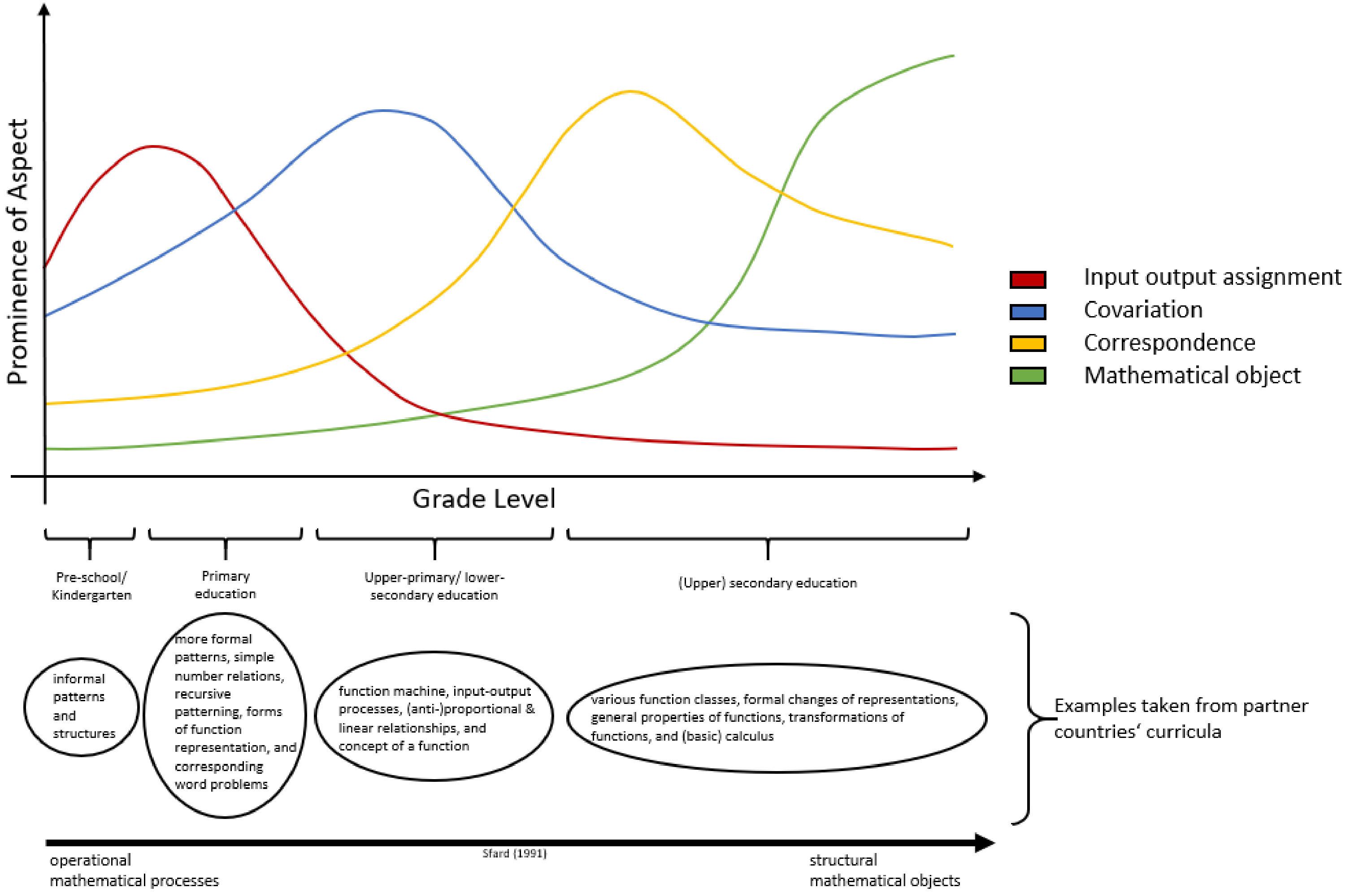
Aspects of functional thinking in the curriculum

	primary	low-secondary	high-secondary (continue with job)	high-secondary (continue with university without mathematics)	high-secondary (continue with university with mathematics)
Input-output					
Covariation					
Correspondence					
Object					

Aspects of functional thinking in the curriculum

	primary	low-secondary	high-secondary (continue with job)	high-secondary (continue with university without mathematics)	high-secondary (continue with university with mathematics)
Input-output	7				
Covariation	2				
Correspondence	2				
Object	1				

Aspects of functional thinking in the curriculum



Aspects of functional thinking in the curriculum

Explain four aspects of the functional thinking on the following functions:

A: $y = \sin x, x \in \mathbb{R}$

B: $y = \cos x, x \in \mathbb{R}$

Representations in the curriculum

	primary	low-secondary	high-secondary (continue with job)	high-secondary (continue with university without mathematics)	high-secondary (continue with university with mathematics)
Graph					
Formula					
Table					
Verbal description					
Other					

Representations in the curriculum

	primary	low-secondary	high-secondary (continue with job)	high-secondary (continue with university without mathematics)	high-secondary (continue with university with mathematics)
Graph	I	P	P	P	P
Formula	I	I - P	P	P	P
Table	P	P	P	P	P
Verbal description	I	I - P	P	P	P
Other	Chain, nomogram, Venn diagrams, ...				

Representations

Represent the function $y = 5x + 10, x \in \mathbb{R}$ in as many ways as possible.
A: Explain, how different representations pinpoint different aspects of functional thinking.
B: Discuss, what are the benefits and restrictions of these representations.

Graph - propedeutics

- **What is needed to be able to use the graphical representation correctly?**

- **Developing the concept of number**
  - $\mathbb{N} \longrightarrow \mathbb{Q}^{+} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{R} \longrightarrow \mathbb{C}$
  - When is the number line "full"?

- **Coordinate system**
  - propedeutics - coordinates in the chessboard and filling in the table
  - problems with the order:
    - **in the chessboard A1 and 1A are the same square**
    - **in a table, many people prefer the row (y-coordinate) over the column (x-coordinate)**
  - a square in the chessboard/table vs. a point in the coordinate system

Formula - propedeutics

What does it take to know how to use a formula correctly?

- **Different understandings of the letter in mathematics**

1. **The letter represents one specific** value
   denotes an unknown number, which we determine by logical reasoning or guessing (not using equivalent equation modifications), for example What number should we substitute after x to make the equation hold: x-5=12.
2. **We don't need to** know **the value of the letter**
   For example, in the problem: For the numbers x and y, x+y=10. What is x+y-8=?
3. **Letter as a designation**
   is an agreed designation, a naming convention, e.g. the letters a, b, c denote the sides of a triangle, the letter t denotes a tone, and so on. For example, 3f=1y, where f denotes feet and y denotes yards.
4. **Letter as one unknown** value
   a letter represents one specific number, we often talk about an unknown that can occur in equations or when modifying expressions, for example 3x-5=4x. When working with the letter in this context, for example, we use a variety of methods for modifying equations, including equivalent modifications.

Formula - propedeutics

What does it take to know how to use a formula correctly?

- **Different understandings of the letter in mathematics**

5. **The letter takes more than one value (any value from the definition field**)
   a letter here denotes several numbers (even infinitely many), as in the inequality $4x$-5<15. Meanwhile, when working with a letter at the level of a letter as an unknown, for example when modifying equations, we often use a letter in this context.
6. **Letter as a constant**
   e.g. π, e.
7. **Letter as parameter**
   for example, the letters k and q in the notation of the general prescription of the linear function $y=kx+q$
8. **Letter as a variable**
   In this context, pupils encounter the letter in the context of the concept of function, which also appears implicitly in school mathematics before its introduction, for example, in the context of exploring the dependence of path on time.

Thematic area of functions in the curriculum

primary	low-secondary	high-secondary (continue with job)	high-secondary (continue with university without mathematics)	high-secondary (continue with university with mathematics)

Thematic area of functions in the curriculum

primary	low-secondary	high-secondary (continue with job)	high-secondary (continue with university without mathematics)	high-secondary (continue with university with mathematics)
Propedeutics	Direct and indirect proportionality Linear function			

Thematic area of functions in the curriculum

- **Which properties of a function or which concepts can be discussed for which functions?**

Linear function	Monotony Inverse function
Power functions	Evenness / Oddness
Goniometric functions	Periodicity
Exponential, Logarithmic	Monotony Inverse function

Functions in other areas of the mathematics curriculum

primary	low-secondary	high-secondary (continue with job)	high-secondary (continue with university without mathematics)	high-secondary (continue with university with mathematics)

Functions in other curriculum areas

- **Geometric formulas (perimeters, contents, ...) as functions**
  - Creating tables
  - Noticing differences and similarities between tables (the content of a square grows faster than its content)
- **Conformity and similarity of formations**
  - Ratio and direct proportionality
- **Analytical equations of some geometric figures**
  - Quadratic function - parabola, Linear function - straight line, ...
- **Equations and inequalities**
  - Graphical solutions of equations and inequalities
  - Using properties of functions in reasoning about modifications (change of sign because the function is decreasing; non-equivalence of the modification because the function is not simple, ...)
- **Statistical models**
  - Linear function as a regression line

Where can you see functions?

You have **10 minutes** to walk across the university building and write down a list of all observed functional dependencies. The group with the longest and at the same time meaningful list wins!

Functions in other subjects - examples of applications

primary	low-secondary	high-secondary (continue with job)	high-secondary (continue with university without mathematics)	high-secondary (continue with university with mathematics)

Functions in other subjects - low-secondary

- **Chemistry:**
  - calculate the mass fraction of a component in solution; the mass of solute, solvent and solution
  - carry out experiments to measure thermal changes in chemical reactions, record the results of the experiments in tables and interpret them,
- **Geography:**
  - describe the apparent path of the sun and moon in the sky (pictures, sketches),
  - determine from the map of time zones where on earth there are more hours than in Slovakia and where there are fewer,
  - identify a selected location on a map using geographical coordinates
  - compare distances on maps of different graphical scales,
  - talk about balloon travel from the equator to the polar countries -> summarize changes in the air with increasing altitude

Functions in other subjects -low-secondary

- **Physics:**
  - analyse graphs, explain the graph progression, process the measured values in the form of a graph (in general)
  - construct a graph of the linear dependence of the trajectory on time for a uniform rectilinear motion
  - construct a graph of the constant velocity versus time for uniform straight-line motion
  - construct a graph of the direct proportionality between current and voltage from the measured values
  - linear dependence of the gravitational force and the mass of the body
  - graphical representation of velocity and trajectory over time
  - constructing a graph of electric current versus electric voltage
- **Informatics:**
  - information in tables, cell, relationships between cells, graphs

Functions in other subjects - high-secondary

- **Chemistry:**
    - solve problems to calculate the mass fraction, and the concentration of a component,
    - sort a group of elements into elements with small and large electronegativity values based on their location in the periodic table
    - Calculate the mass of the reactant or product based on the chemical equation, given the mass of the solid product or reactant,
    - determine the value of the heat of reaction of the reverse reaction based on the value of the heat of reaction of the direct reaction using the 1st thermochemical law
    - compare the rate of chemical reactions by observation
    - Explain the nature of the effect of changes in temperature, reactant and catalyst concentration on the rate of a chemical reaction,
    - explain the nature of the effect on the equilibrium state of a system of adding a reactant or removing a product, changing temperature and pressure

Functions in other subjects - high-secondary

- **Geography:**
    - determine the location of any place on the map using geographical coordinates
    - retrieve and interpret statistics, facts and relevant facts from reliable information sources
    - creatively use geographical knowledge in various graphic forms (content of thematic maps, tables, diagrams, charts, cartograms, carto-diagrams)
    - understand and use adequately the data presented in GPS instruments and navigators
    - calculate the actual distance of places on a map from a numerical or graphical scale
    - correctly interpret climate data presented in various graphical and textual forms (tables, charts, graphs, climate diagrams, thematic maps),
    - know the meaning and reliability of meteorological forecasts,
    - compare the influence of internal and external geological processes on the formation of the Earth's surface
    - correctly interpret information on population development and composition presented in the form of graphs, tables, age pyramids, diagrams and thematic maps,
    - correctly interpret statistical data and economic indicators of the economic performance of countries and regions of the world
    - correctly interpret statistical data and economic indicators of the economic performance of individual regions, states
    - correctly interpret information on the development and composition of the region's population presented in the form of graphs, tables, age pyramids, diagrams and thematic maps
    - correctly interpret information on the development and composition of the population of Slovakia, presented in the form of graphs, tables, age pyramids, diagrams and thematic maps
    - estimate the development and possible risks of changes in key industries in Slovakia,

Functions in other subjects - high-secondary

- **Physics:**
  - analysis of the graphical representation of the dependence of the trajectory on time (uniform, non-uniform, decelerated and accelerated movements)
  - interpret the slope of a graph of linear dependence and the intersections of the graph with the coordinate axes
  - display of the work done in a force versus displacement graph
  - free fall - constructing a graph of path versus time, determining the dependence of speed on time
  - temperature dependence of electron motion, temperature dependence of electrical resistance
  - linear dependence, linear dependence graph
  - temperature dependence of metallic conductor resistance
  - harmonic oscillatory motion (the time diagram of the oscillatory motion is a sinusoid or cosine-sinusoid)

Functions in other subjects - high-secondary

- **Informatics:**
  - analyze the problem, propose an algorithm for solving the problem, write the algorithm in an understandable formal form, verify the correctness of the algorithm
  - understand the finished programs, determine the properties of inputs, outputs and relationships between
  - them
  - solve tasks using commands with various constraints on the use of commands,
  - variables, types and operations
  - input and output of information depending on its type
  - spreadsheet calculator - formula, function
  - input and output devices
  - problem, algorithm, algorithms from everyday life, ways of writing an algorithm
  - programming language - commands (assignment, input, output), variables

Enhancing functional thinking from primary to upper secondary school

Co-funded by the
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funThink
Enhancing functional thinking
from primary to upper secondary school

Forster students‘ functional thinking

Introduction to learning environments

Conference, Place

Date

Agenda

- **Design principles**
  - Inquiry-based learning
  - Situatedness
  - Embodiment
  - (Digital) tools
- **Learning environments**
  - General information
  - Exploration of a learning environment

Learning objectives:

- Learn about the four design principles (more later).
- Know the structure of the learning environments with handouts and teacher guide
- Identify the learning objectives of the learning environments and link them to the basic concepts of functional thinking

Design principles of the learning environments

Overview

INQUIRY-BASED LEARNING

SITUATEDNESS

EMBODIMENT

(DIGITAL) TOOLS

Design principles of the learning environments

Enhancing functional thinking
from primary to upper secondary school

1. Inquiry-based learning

https://primas-project.eu/wp-content/uploads/sites/323/2017/11/primas_final_publication.pdf [15.6.2022]

Valued outcomes

- Inquiring minds: critical & creative
- Preparation for uncertain future & lifelong learning
- Understanding of nature of science & math

Teachers

- Foster and value students' reasoning
- From telling to supporting and scaffolding
- Connect to students' experiences

Classroom culture

- Shared sense of ownership and purpose
- Value mistakes, contributions (open-minded)
- Dialogic

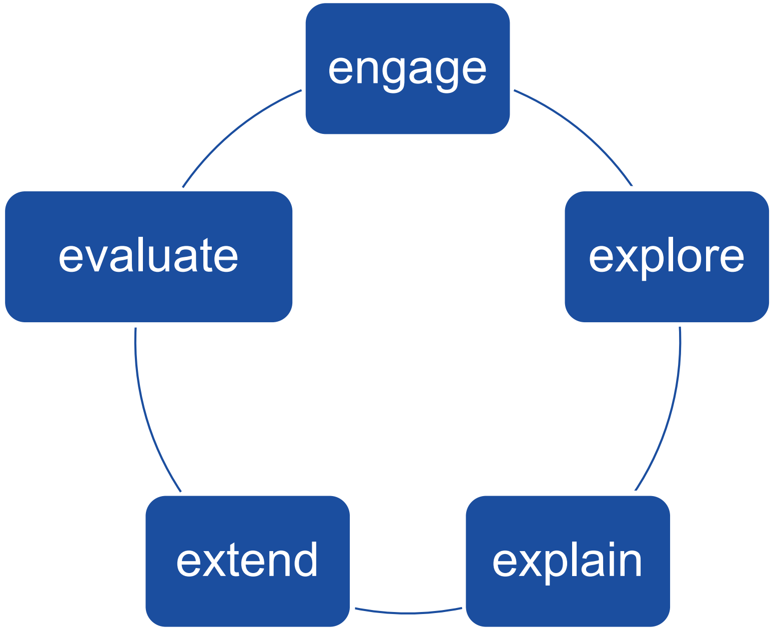
Students

- Pose questions
- Inquire / 5 E's: Engage, explore, explain, extend and evaluate
- Collaborate

Learning Environment

- Problems: Open, multiple solution strategies, experienced as real and/or scientifically relevant
- Access to tools and resources
- From problems to explanations (not from examples to practicing)

Central activities:


engage
explore
explain
extend
evaluate

(for more on this, see e.g. Dorier & Maass, 2020; Artigue & Blomhøj, 2013).

Support of:

- Curiosity & critical thinking
- Construction & application of new knowledge
- Cooperation & Communication

→ Students formulate questions and answer them

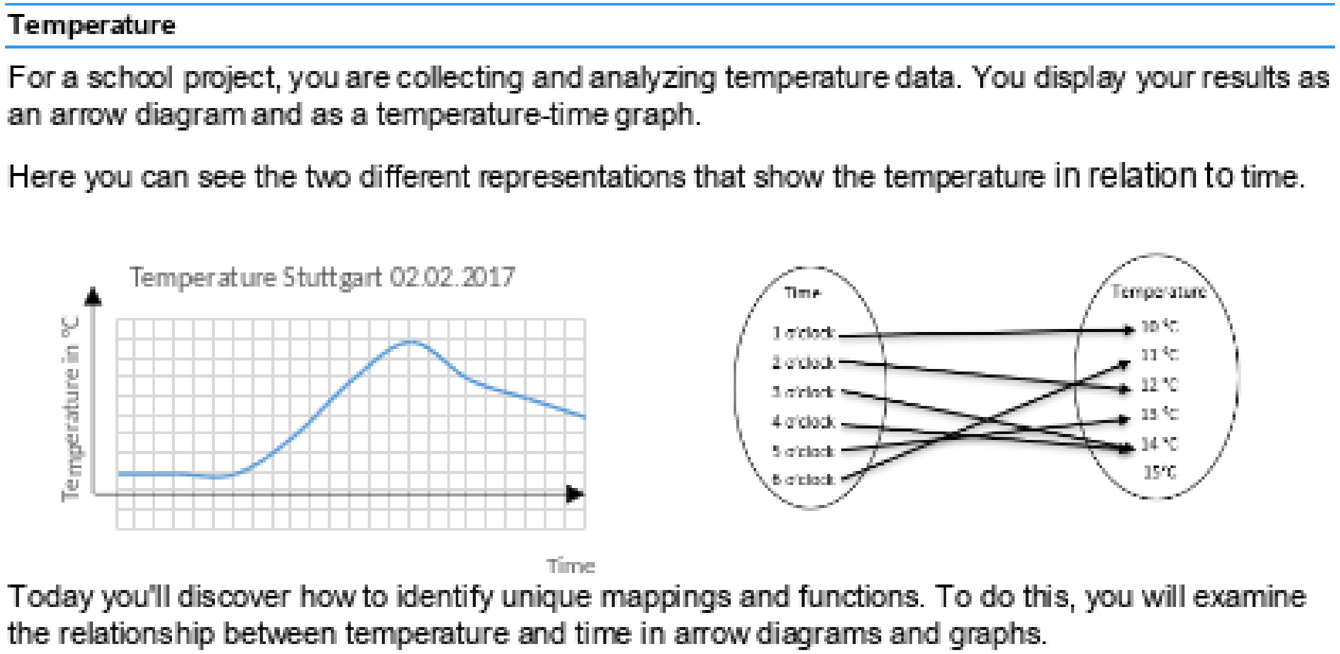
Design principles of the learning environments

2. Situatedness

Fundamental for function concept:

Identify, create & reproduce dependencies or relationships between variables of the physical, social and / or spiritual world.

(Freudenthal, 1983, p. 494)



Temperature

For a school project, you are collecting and analyzing temperature data. You display your results as an arrow diagram and as a temperature-time graph.

Here you can see the two different representations that show the temperature in relation to time.

Today you'll discover how to identify unique mappings and functions. To do this, you will examine the relationship between temperature and time in arrow diagrams and graphs.

(for more on this, see e.g. Freudenthal, 1983; Gravemeijer & Terwel, 2000).

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from primary to upper secondary school

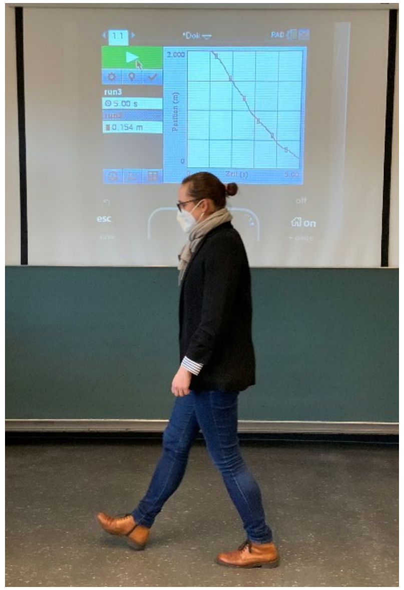
Design principles of the learning environments

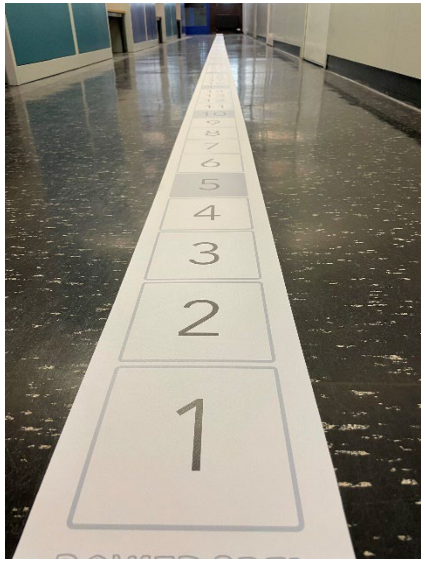
3. Embodiment

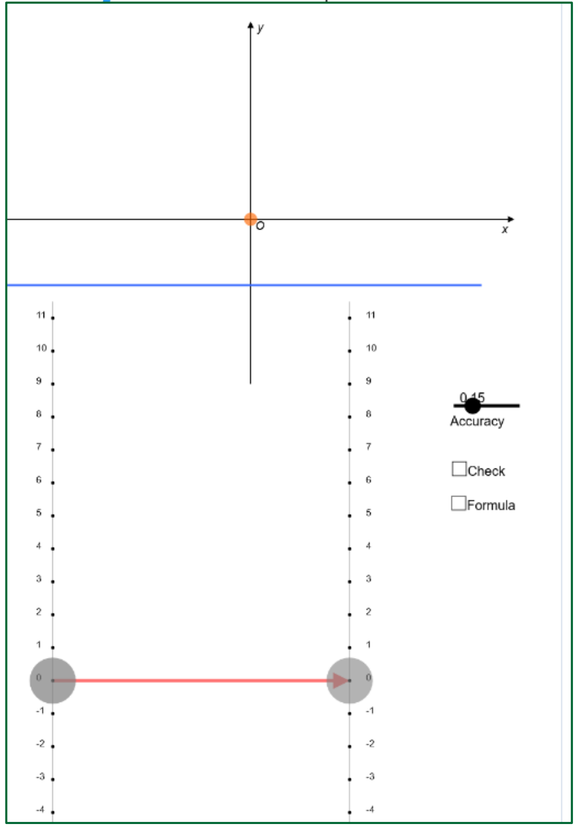
Central ideas:

- Everything that a person (consciously) experiences and perceives becomes part of cognition
- Physical experiences are essential for cognition
- Mathematical understanding processes can build on physical experiences / movements

(for more on this, see e.g. Duijzer et al., 2019; Lakoff & Nunez, 2000).







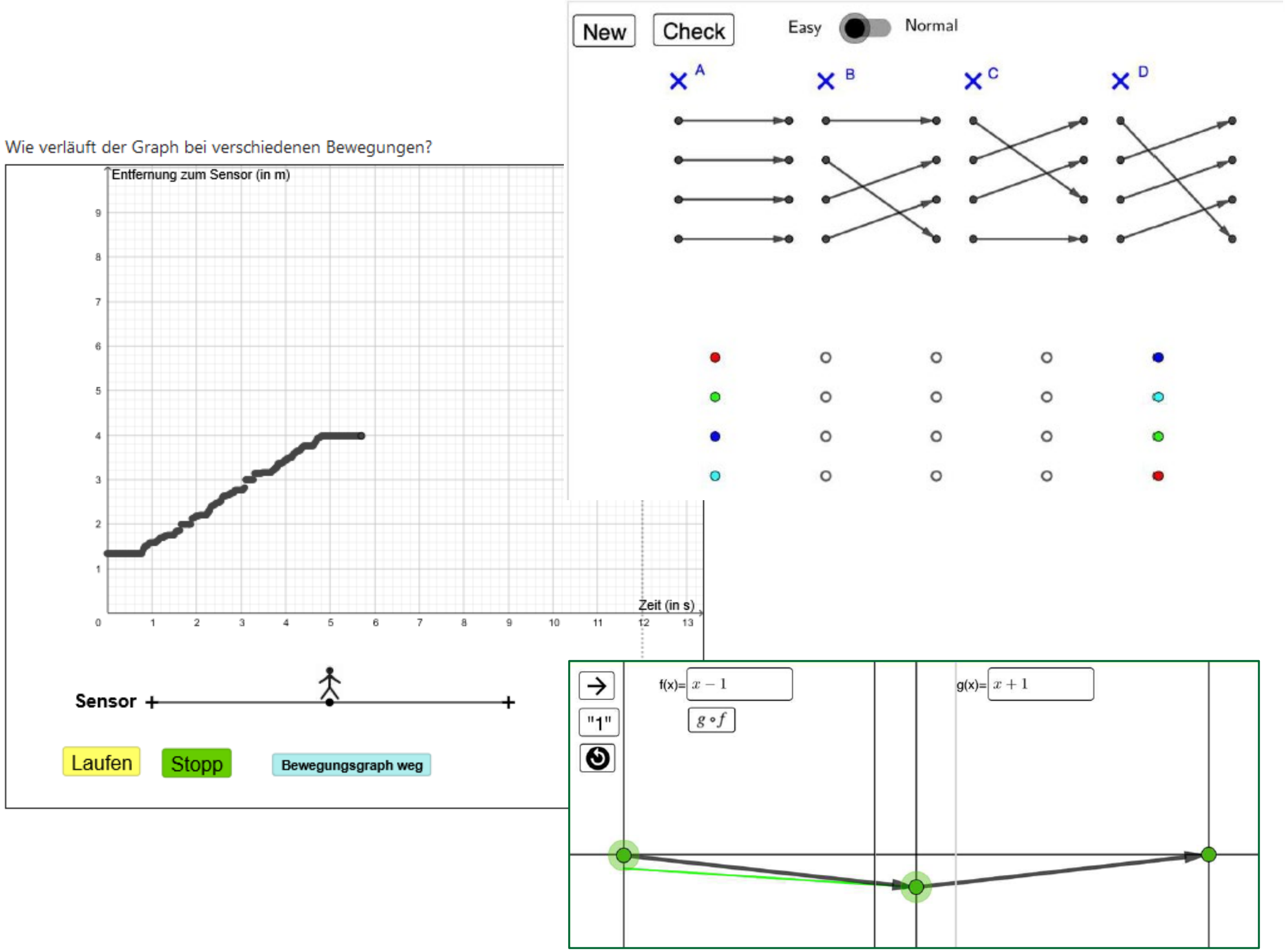
Design principles of the learning environments

4. (Digital) tools

Central ideas:

- Use of tools expands scope of action & allows many (cognitive) tasks to be completed more easily and efficiently
- Variety of digital tools available for mathematics education
- Combintation with embodiment possible

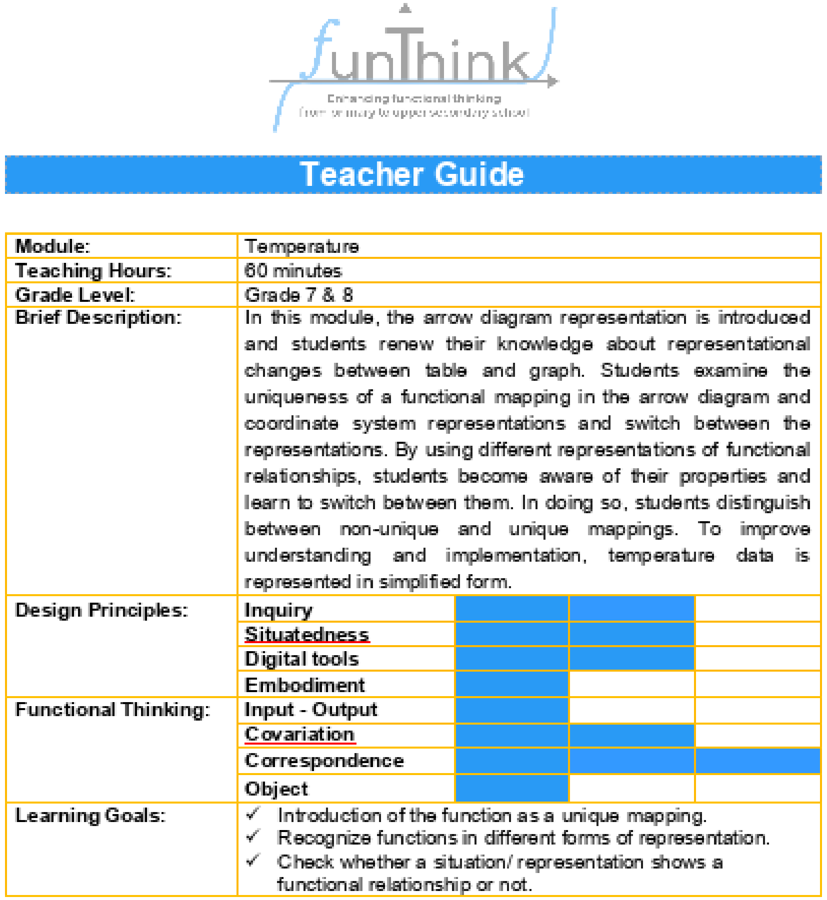
(for more, see e.g. Drijvers, 2019; Hoyles, 2018; Monaghan et al., 2016).


Wie verläuft der Graph bei verschiedenen Bewegungen?

Introduction to the learning environments

Components of the learning environments:

- Teacher guide
    - ➢ First page(s)
        - ✓ Module name
        - ✓ Time required
        - ✓ Target group
        - ✓ Module description
        - ✓ Focus of the design principles
        - ✓ Aspects of Functional Thinking
        - ✓ Learning objectives



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Enhancing functional thinking from primary to upper secondary school

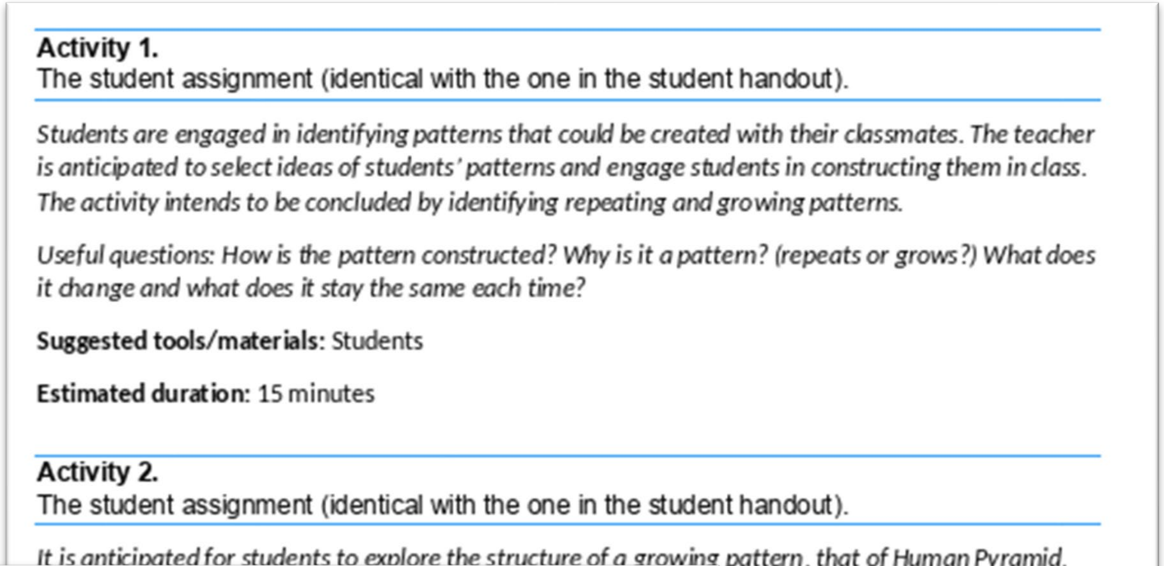
Teacher Guide

Module:	Temperature			
Teaching Hours:	60 minutes			
Grade Level:	Grade 7 & 8			
Brief Description:	In this module, the arrow diagram representation is introduced and students renew their knowledge about representational changes between table and graph. Students examine the uniqueness of a functional mapping in the arrow diagram and coordinate system representations and switch between the representations. By using different representations of functional relationships, students become aware of their properties and learn to switch between them. In doing so, students distinguish between non-unique and unique mappings. To improve understanding and implementation, temperature data is represented in simplified form.			
Design Principles:	**Inquiry**	🟦	🟦	
	Situatedness	🟦	🟦	
	Digital tools	🟦	🟦	
	Embodiment	🟦		
Functional Thinking:	**Input - Output**	🟦		
	Covariation	🟦	🟦	
	Correspondence	🟦	🟦	🟦
	Object	🟦		
Learning Goals:	✓ Introduction of the function as a unique mapping. ✓ Recognize functions in different forms of representation. ✓ Check whether a situation/ representation shows a functional relationship or not.			

Introduction to the learning environments

Components of the learning environments:

- Teacher guide
  - First page(s)
  - Lesson plan with further materials
  - Tasks with explanations (sample solutions)
  - If necessary, further materials
  - Didactic notes



Activity 1.
The student assignment (identical with the one in the student handout).

Students are engaged in identifying patterns that could be created with their classmates. The teacher is anticipated to select ideas of students' patterns and engage students in constructing them in class. The activity intends to be concluded by identifying repeating and growing patterns.

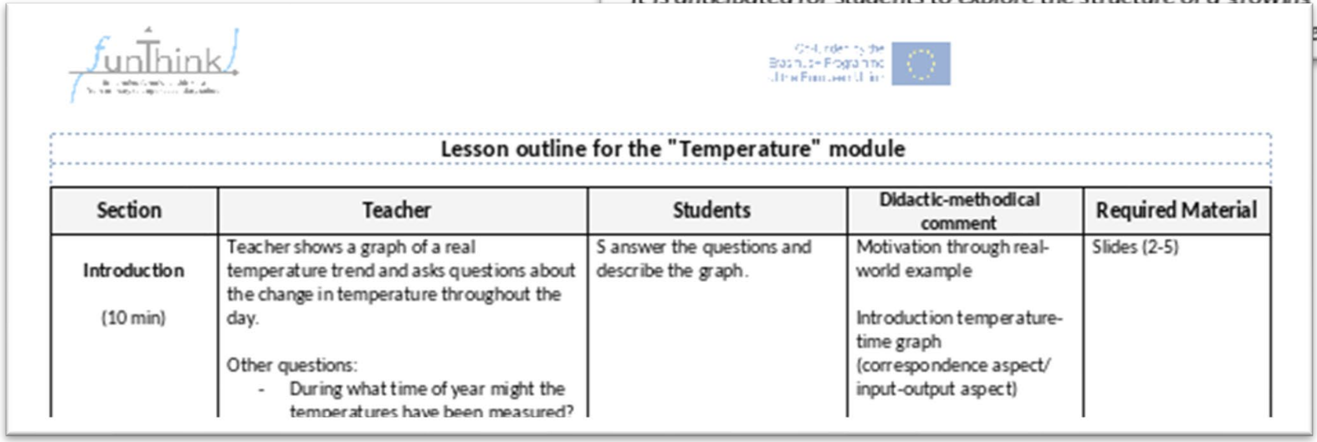
Useful questions: How is the pattern constructed? Why is it a pattern? (repeats or grows?) What does it change and what does it stay the same each time?

Suggested tools/materials: Students

Estimated duration: 15 minutes

Activity 2.
The student assignment (identical with the one in the student handout).

It is anticipated for students to explore the structure of a growing pattern, that of Human Pyramid,
e "next" one). Students could be



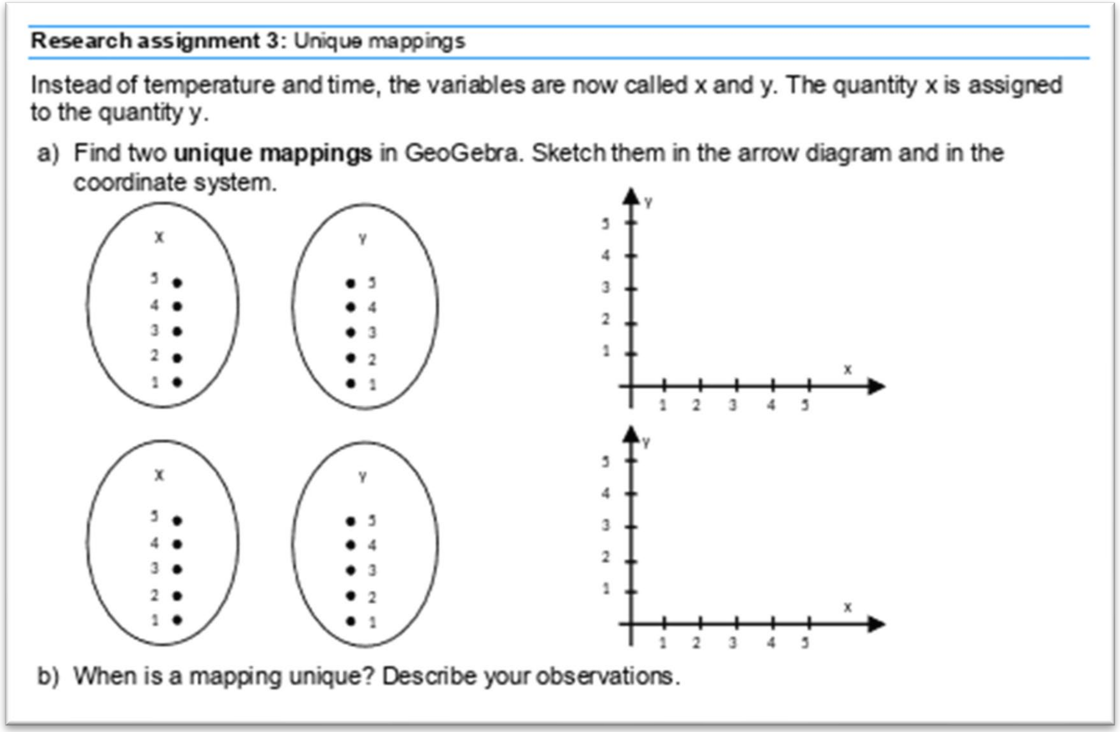
Lesson outline for the "Temperature" module

Section	Teacher	Students	Didactic-methodical comment	Required Material
Introduction (10 min)	Teacher shows a graph of a real temperature trend and asks questions about the change in temperature throughout the day. Other questions: - During what time of year might the temperatures have been measured?	S answer the questions and describe the graph.	Motivation through real-world example Introduction temperature-time graph (correspondence aspect/ input-output aspect)	Slides (2-5)

Introduction to the learning environments

Components of the learning environments:

- Teacher guide
- Student handout

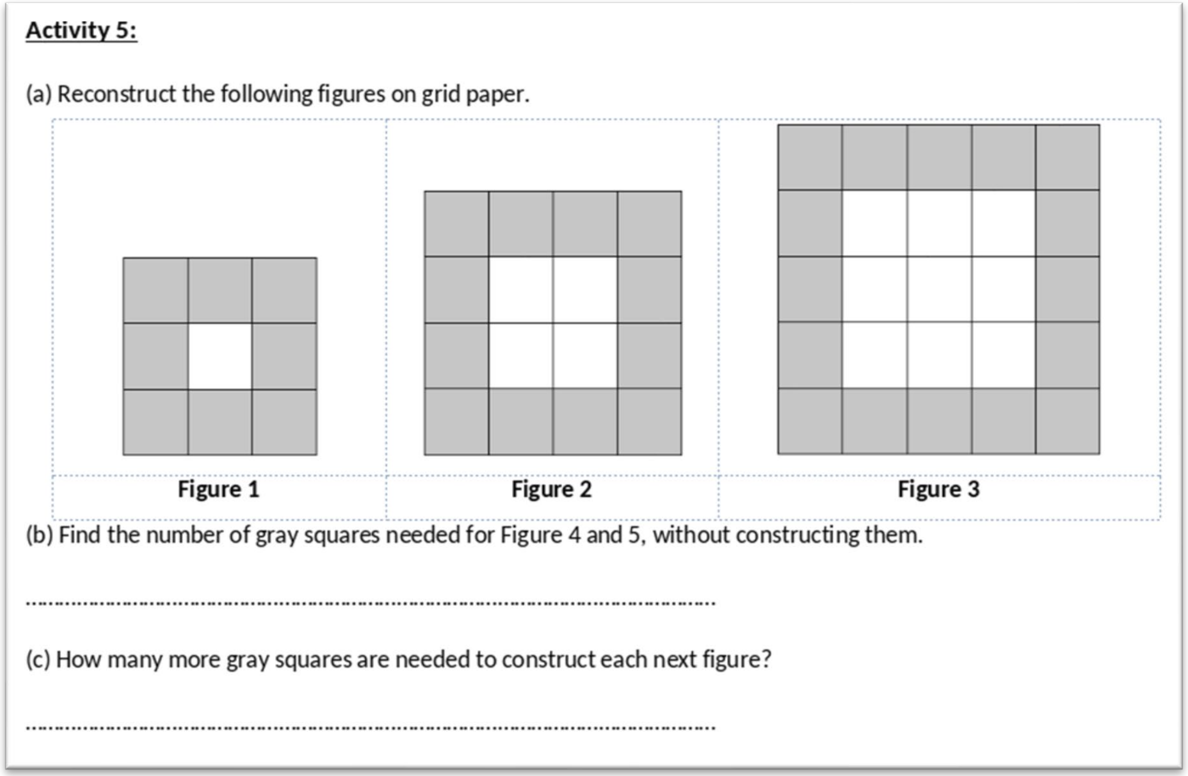


Research assignment 3: Unique mappings

Instead of temperature and time, the variables are now called x and y. The quantity x is assigned to the quantity y.

a) Find two **unique mappings** in GeoGebra. Sketch them in the arrow diagram and in the coordinate system.

b) When is a mapping unique? Describe your observations.



Activity 5:

(a) Reconstruct the following figures on grid paper.

Figure 1 **Figure 2** **Figure 3**

(b) Find the number of gray squares needed for Figure 4 and 5, without constructing them.

……………………………………………………………………………………………………

(c) How many more gray squares are needed to construct each next figure?

……………………………………………………………………………………………………

Chosen Learning Environment

Activity (40 minutes partner work):
Getting to know the chosen learning environment.

- Familiarise yourself with the teacher guide.
- Work on the tasks in the handout.
- Answer the following questions:
  - What prior knowledge do you expect for the learning environment?
  - What are the learning objectives of this learning environment / the different tasks?
  - Which aspects of functional thinking are addressed in this module?
  - What forms and changes of representation are focused on in this learning environment?

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Getting to know the chosen learning environment.

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  - ➢ What are the learning objectives of this learning environment / the different tasks?
  - ➢ Which aspects of functional thinking are addressed in this module?
  - ➢ What forms and changes of representation are focused on in this learning environment?

Discussion / Reflection

Chosen Learning Environment

Activity (small group work):
Create further tasks to match the chosen learning environment.
Provide details of:

- Learning objective(s)
- Addressed aspects of functional thinking
- Representations used or changes in representation

Thank you for your attention and until next time!

Co-funded by the Erasmus+ Programme of the European Union

Literature

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Lakoff, G. & Núñez, R. E. (2000). Where mathematics comes from: How the embodied mind brings mathematics into being. Basic Books.

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This material is provided by the FunThink Team, responsible institution: Ludwigsburg University of Education

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Forster students‘ functional thinking

Inquiry-based learning as a design principle

Date

Agenda

- **Inquiry-based learning**
  - *As a design principle*
  - *Example of implementation*

Learning objectives:

- ➢ Familiarization with the design principle inquiry-based learning in the context of different tasks
- ➢ Explore and reflect possible implementations in learning environments

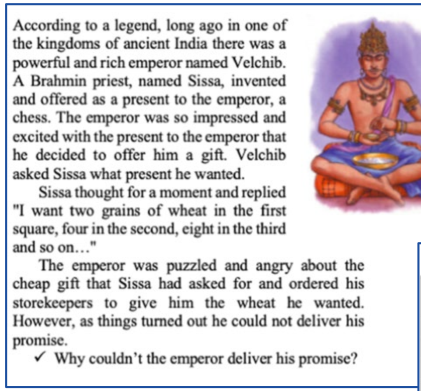
Inquiry-based learning

Activity (handout):

Below you see three different approaches from textbooks to introduce powers.

1. Compare the different didactical approaches. In particular, consider the extent to which the examples incorporate students' prior knowledge.
2. Which example would you choose? Provide reasons.

①

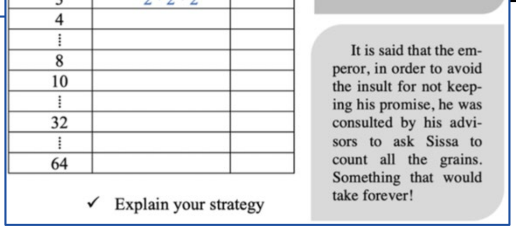


According to a legend, long ago in one of the kingdoms of ancient India there was a powerful and rich emperor named Velchib. A Brahmin priest, named Sissa, invented and offered as a present to the emperor, a chess. The emperor was so impressed and excited with the present to the emperor that he decided to offer him a gift. Velchib asked Sissa what present he wanted.

Sissa thought for a moment and replied "I want two grains of wheat in the first square, four in the second, eight in the third and so on…"

The emperor was puzzled and angry about the cheap gift that Sissa had asked for and ordered his storekeepers to give him the wheat he wanted. However, as things turned out he could not deliver his promise.

✓ Why couldn't the emperor deliver his promise?



Fill in the table:

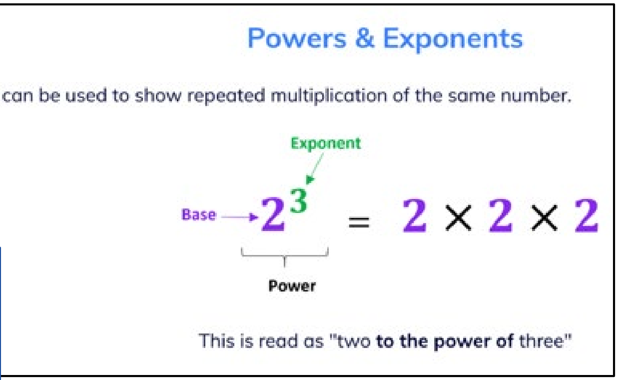
Square	Number of wheat grains	Result
1	2	2
2	$2 \cdot 2$	4
3	$2 \cdot 2 \cdot 2$	
4		
⋮		
8		
10		
⋮		
32		
⋮		
64		

✓ Explain your strategy

To produce this huge quantity of grains, which is actually a 20 - digit number, one has to plant the whole Earth 76 times!

It is said that the emperor, in order to avoid the insult for not keeping his promise, he was consulted by his advisors to ask Sissa to count all the grains. Something that would take forever!

②



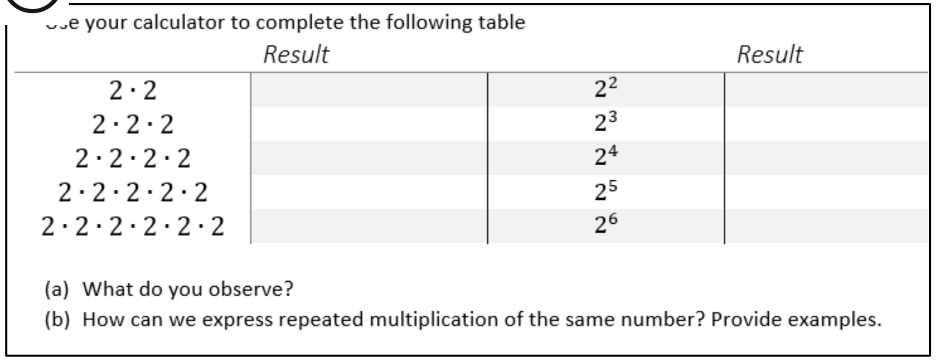
Powers & Exponents

Powers can be used to show repeated multiplication of the same number.

$$2^3 = 2 \times 2 \times 2$$

This is read as "two **to the power of** three"

③



Use your calculator to complete the following table

	Result		Result
$2 \cdot 2$		2^2	
$2 \cdot 2 \cdot 2$		2^3	
$2 \cdot 2 \cdot 2 \cdot 2$		2^4	
$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$		2^5	
$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$		2^6	

(a) What do you observe?
(b) How can we express repeated multiplication of the same number? Provide examples.

(Athanasiou et al., 2016a, p. 47f & 2016b, p. 17)

Inquiry-based learning

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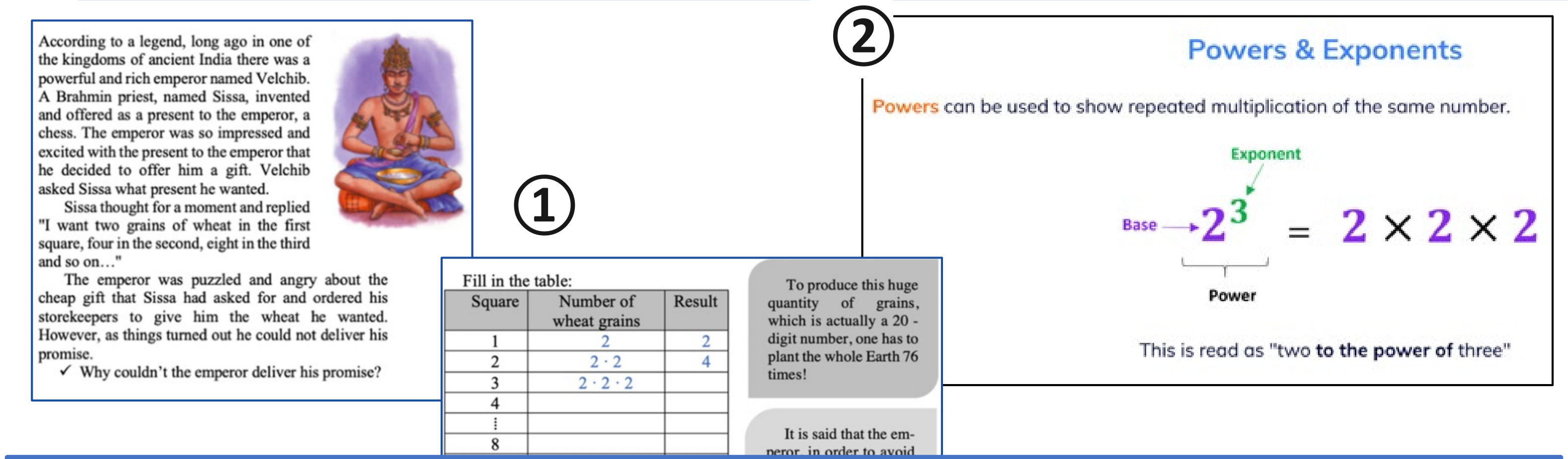

①

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Fill in the table:

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⋮		
8		

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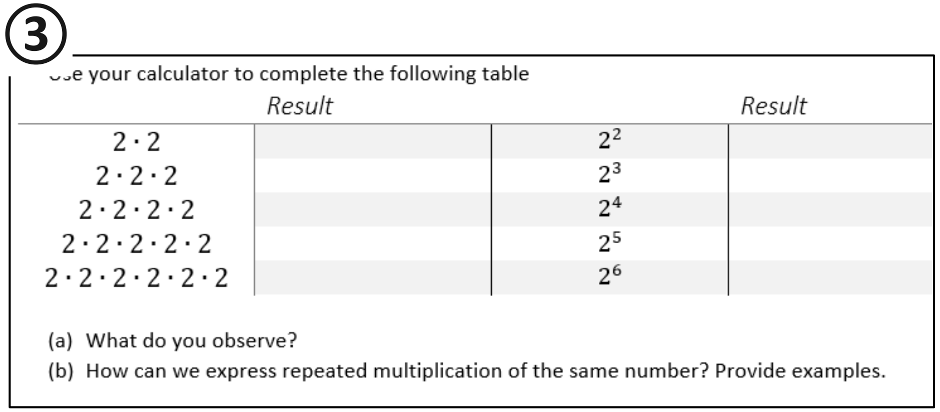

③

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(a) What do you observe?
(b) How can we express repeated multiplication of the same number? Provide examples.

hanasiou et al., 2016a, p. 47f & 2016b, p. 17)

Discussion:
Which approaches enable research-based learning? How?
What characterises inquiry-based learning?

Inquiry-based learning

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Inquiry-based learning in mathematics education (Christou et al., 2023)

- **Base:**
  A question or problem for which answers can be found through exploration or investigation.
- **Context:**
  Problems arise from everyday life, history, art or science
- **Lesson design:**
  - ➢ Consideration of mathematical concepts involved
  - ➢ Inclusion of elements that support research and experimentation
  - ➢ Use of adequate language
  - ➢ Use of appropriate tools

Inquiry-based learning

Role of the teacher

- Supporting students in ...
  - ➢ Formulating questions
  - ➢ Use of prior knowledge
  - ➢ Structuring during the development of knowledge
- Encouraging discussions

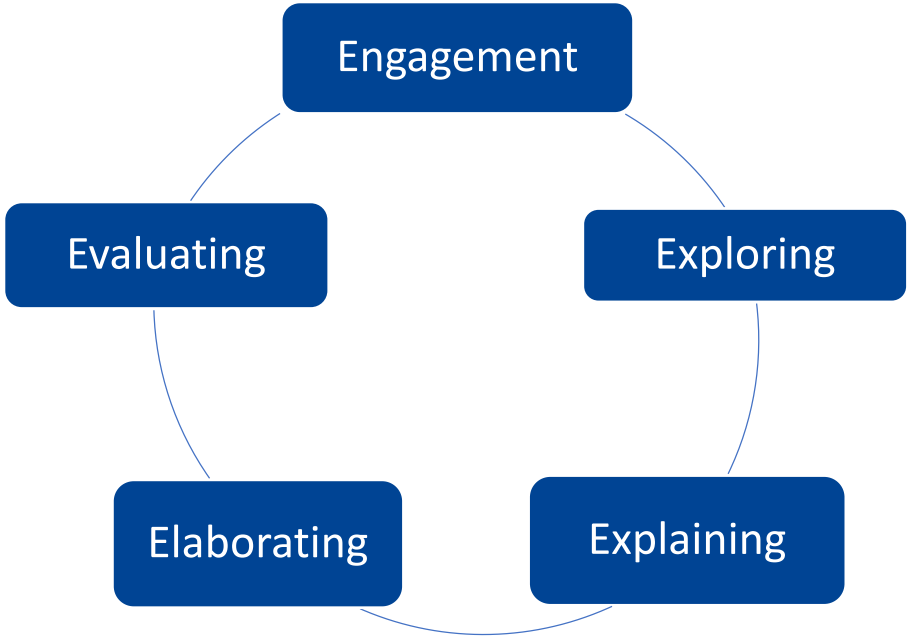
Prerequisite: Tasks that facilitate research-based learning

(Dorier & Maass, 2020)

Inquiry-based learning – IBE (first approach)

- Curiosity and critical thinking
- Constructing and applying new knowledge
- Cooperation and communication

Research cycle 5E



Engagement

Exploring

Explaining

Elaborating

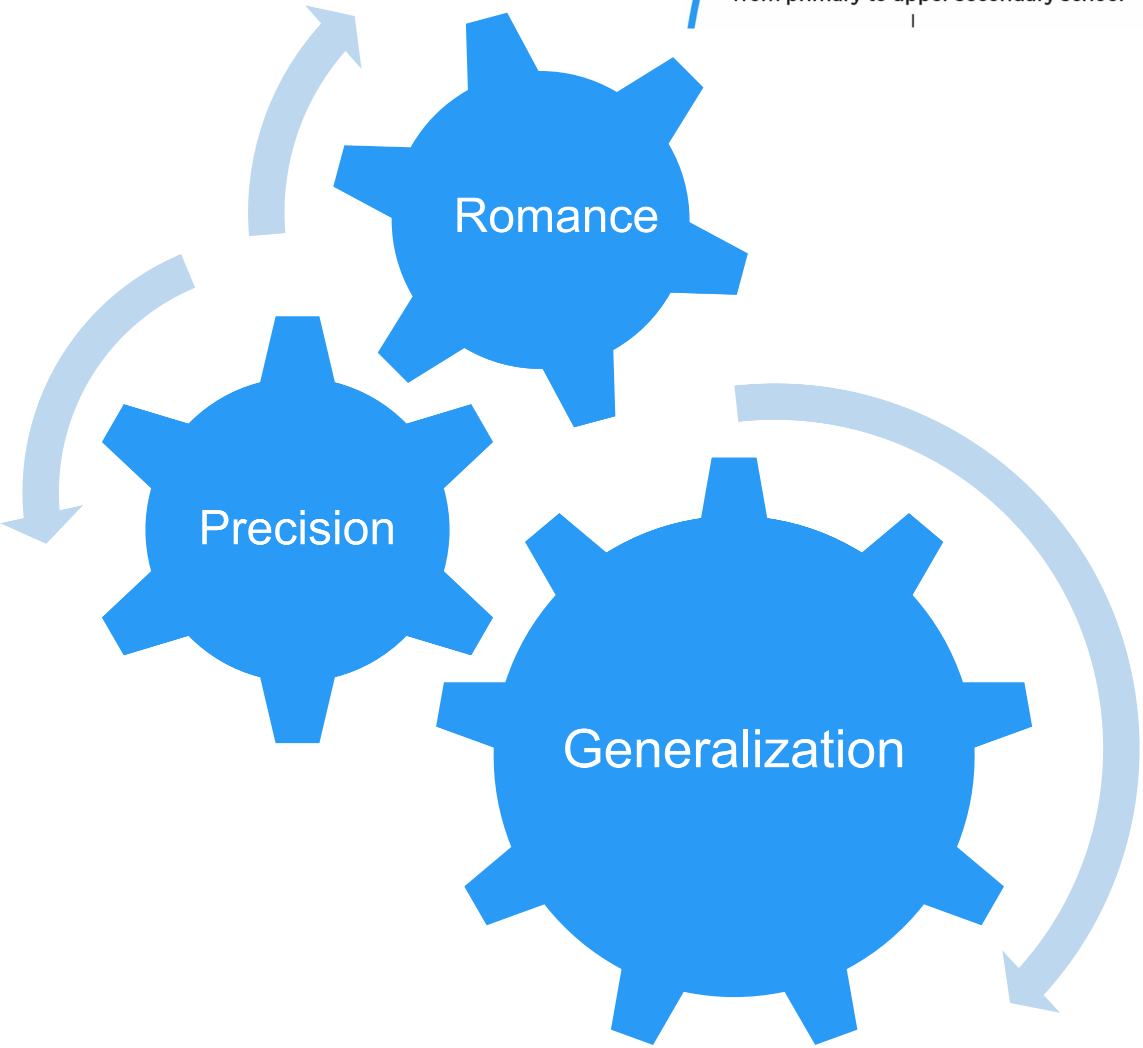
Evaluating

Students formulate questions and seek answers to them

Further information:
Dorier & Maass, 2020; Artigue & Blomhøj, 2013

Inquiry-based learning (second approach)

Phases of Inquiry Learning in Mathematics Education (Whitehead, 1929)



Inquiry-based learning

Phase 1: Romance (Christou et al., 2023)

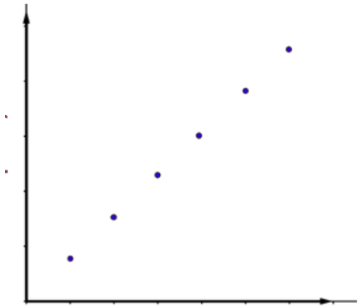
- Introduced with the idea of **exploration**
- Pupils learn about objects, phenomena and ideas in a casual and unstructured way by listening, looking and manipulating (Evans, 1998).
- Curiosity for a new mathematical concept is stimulated in students.
- Interest is maintained, on the basis of which skills and knowledge are to be developed.

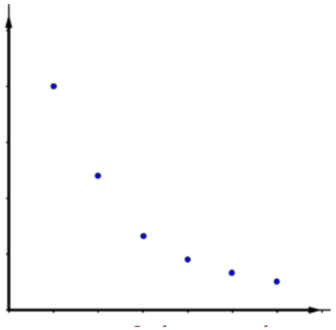
Inquiry-based learning

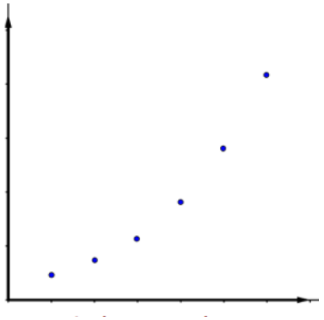
Example of Phase 1: Romance (Athanasiou et al., 2016b, p. 116)

Learning Goal: Students recognise antiproportional relationships in everyday situations.

Task: A factory has a certain number of machines with the same production capacity each that are operated independently. The production manager wants to plan production for a specific order with a fixed delivery date. The person in charge has the option of varying the number of machines producing. Which of the following diagrams shows the time needed to complete the order depending on the number of machines? Explain.







Exploration

Inquiry-based learning

Exploration (Christou et al., 2023)

- Enables lively, exciting learning opportunities based on own experiences
- Promotes understanding of mathematical content by identifying problems and exchanging them
- Encourages students to reflect on / discuss the chosen content / ideas.
  → Exchange / reflections change thinking
- Provides space for students to think about problems and link their own interests to real problems.
  → Encourages curiosity
- Offers students the opportunity to do mathematics, to see the world through mathematical glasses and to "be a mathematician" (cf. Marshman et al., 2011).

Inquiry-based learning

Phase 2: Precision (Christou et al., 2023; Evans, 1998)

- Students develop knowledge and skills needed to develop a new concept
- Includes finding and organizing conceptual structures, rules, etc.
- Is supported by **investigation**
  - ➢ Includes editing of a variety of examples
  - ➢ Starting from the procedure in specific cases, rules are generalized

Inquiry-based learning

Example of exploration in phase 2 (precision)

In GeoGebra you see a rectangle with a constant size of 24cm^2. One of the vertices of the rectangle is the fixed point O (0/0). The opposite point is the point A.
Link: https://www.geogebra.org/m/kzktxmch

Move point A and fill in the table:

Length side a	Length side b	Area rectangle	Coordinates of A (a/b)
1		$24cm^2$	
	12	$24cm^2$	
	8	$24cm^2$	
4		$24cm^2$	
	3	$24cm^2$	
	2	$24cm^2$	

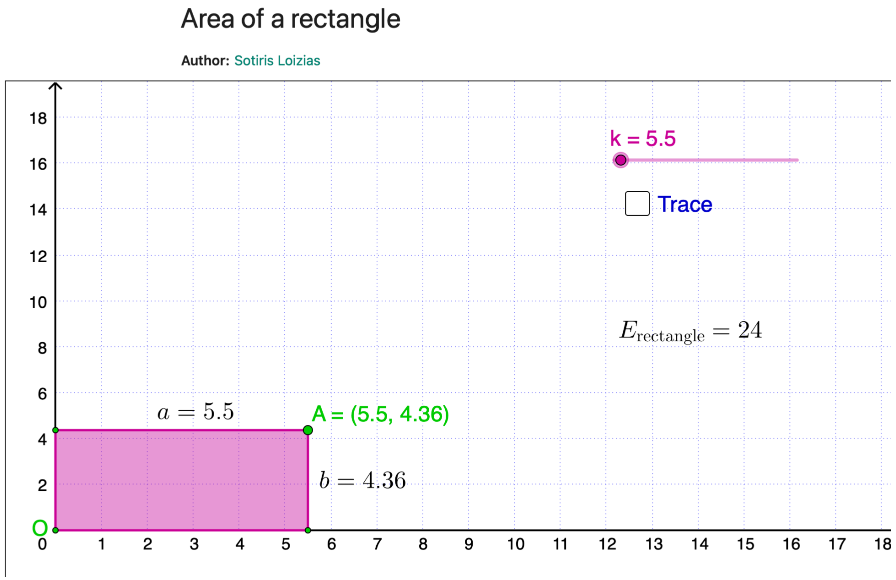
Investigation

Inquiry-based learning

Example of investigation in phase 2 (precision)

Determine the relationship between the x and y coordinates of point A (corresponding to sides a and b of the rectangle):

→ What do you recognize if you look at different values of a and the assigned values of b?



Area of a rectangle

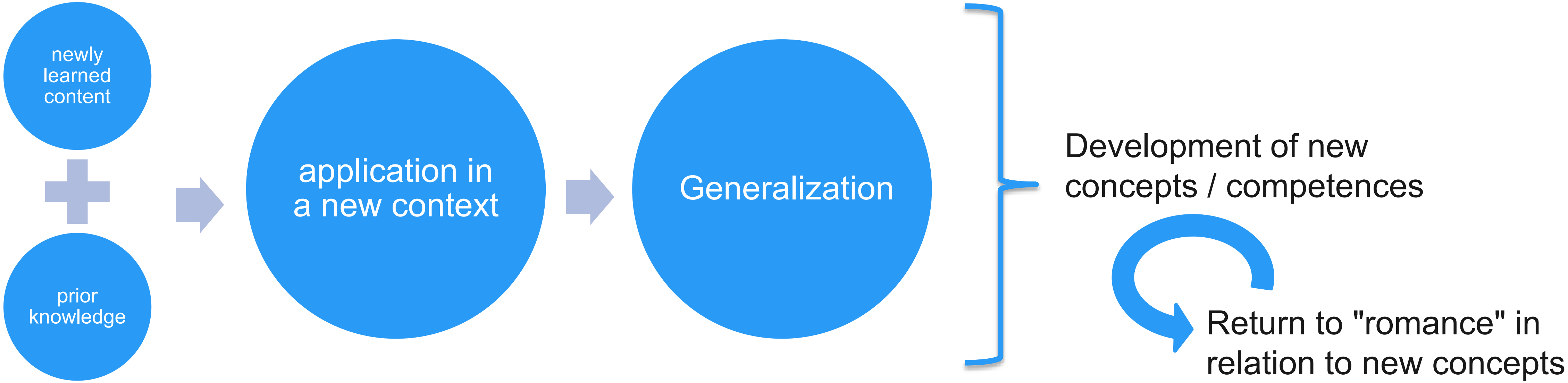
Author: Sotiris Loizias

Investigation

Inquiry-based learning

Phase 3: Generalization (Christou et al., 2023)

- Results in the comprehension-oriented development of a new concept and prevents memorization
- Typical activities: Linking newly learned skills with prior knowledge, applying in different contexts, generalising.



Development of new concepts / competences

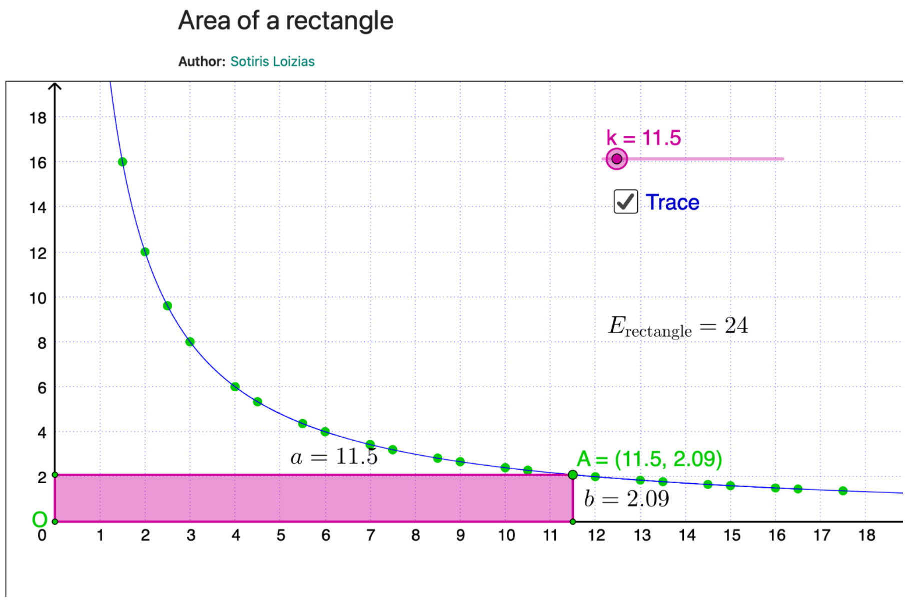
Return to "romance" in relation to new concepts

Inquiry-based learning

Example for phase 3: Generalization

- Move the slider k and observe the change in the coordinates of point A.
- Display the trace by activating trace. Describe the resulting graph.
- Describe the graph of an antiproportional assignment.

→ What is the corresponding function equation?



Area of a rectangle

Author: Sotiris Loizias

Inquiry-based learning

Investigation (Christou et al., 2023)

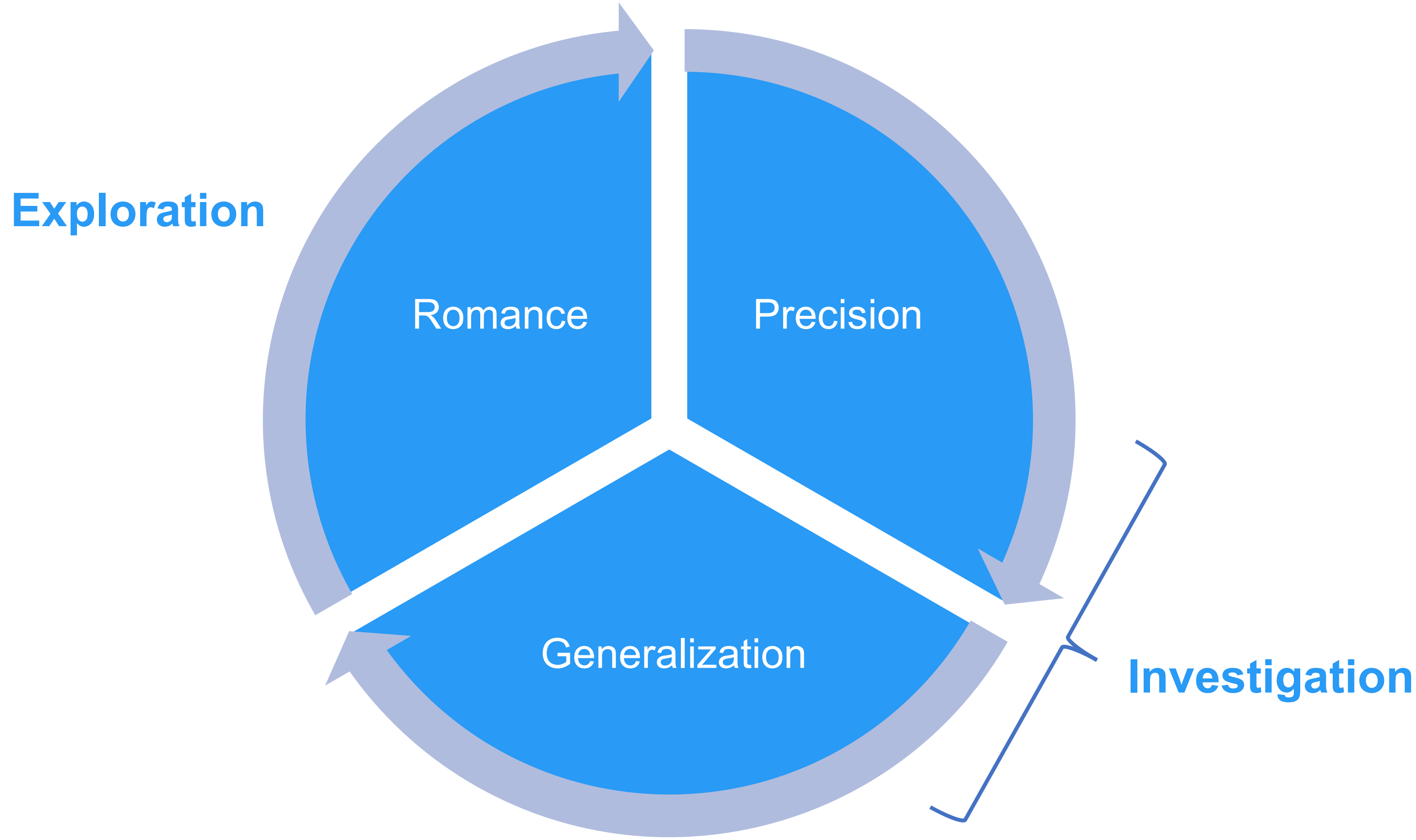
- Refers to an activity that has its origins in mathematics or the real world and is suitable for investigation
- Includes asking questions, collecting data, hypothesising, reflecting and drawing conclusions.

  → These processes take place individually, in small groups and in the class as a whole.
- The curiosity aroused during the exploration phase can be satisfied
- In the course of the investigation, students develop skills that can be applied to other problems.

  → Expanding their knowledge and skills.

Inquiry-based-learning

Learning process in inquiry learning in mathematics education (Christou et al., 2023)



Exploration

Romance

Precision

Generalization

Investigation

Inquiry-based learning

Activity teaching example (15 min individual work / plenary):

Look at the lesson sequence and work through the following questions:

1. To what extent do the activity and discussion support inquiry-based learning? Provide reasons!
2. How could inquiry-based learning be further promoted?

Watch implementation video patterns.

Inquiry-based learning

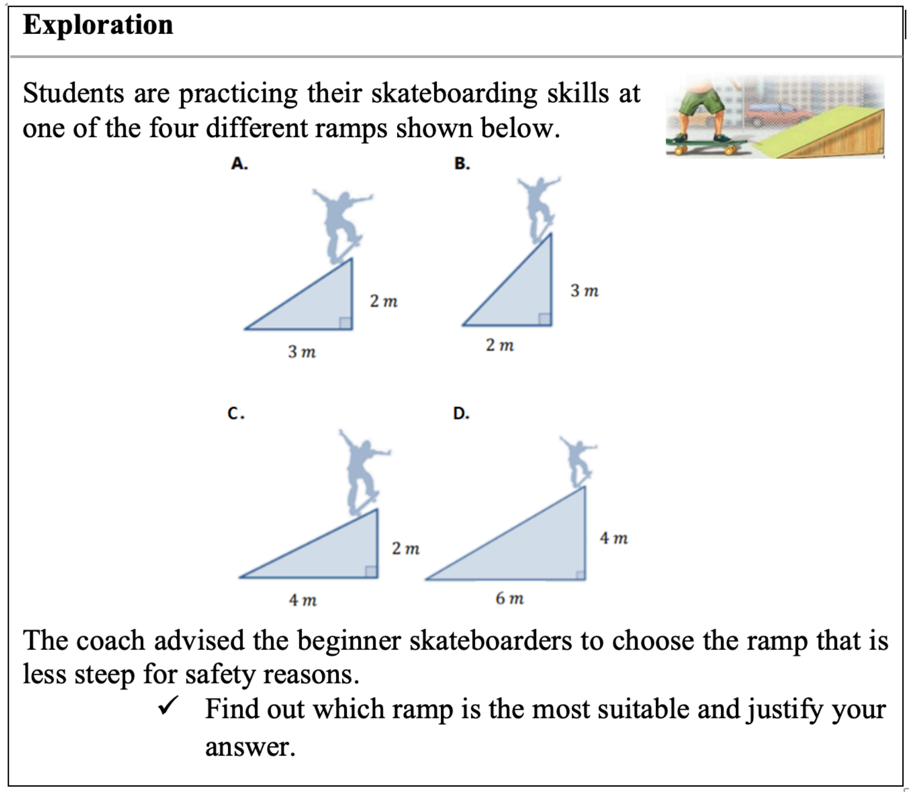
Activity (15 min small group work):
Develop a task that facilitates inquiry-based learning using the following learning objective:

Year 8 pupils understand the slope of a straight line as change. Introduce the slope triangle.

First think about a suitable situation. The following GeoGebra application can be used and adapted:
https://www.geogebra.org/classic/mvvajgsr

Inquiry-based learning

Example: The concept of slope (Athanasiou et al., 2016b, p. 64)



Exploration

Students are practicing their skateboarding skills at one of the four different ramps shown below.

The coach advised the beginner skateboarders to choose the ramp that is less steep for safety reasons.

- ✓ Find out which ramp is the most suitable and justify your answer.

Romance

Inquiry-based learning

Example: The concept of slope (Athanasiou et al., 2016b, p. 65)

✓ Select point *A* and then change the coordinates of point *B*, so that the robot moves one unit to the right (horizontal change). Calculate the vertical shift of the above movement. Consider whether this applies to any starting point *A*.

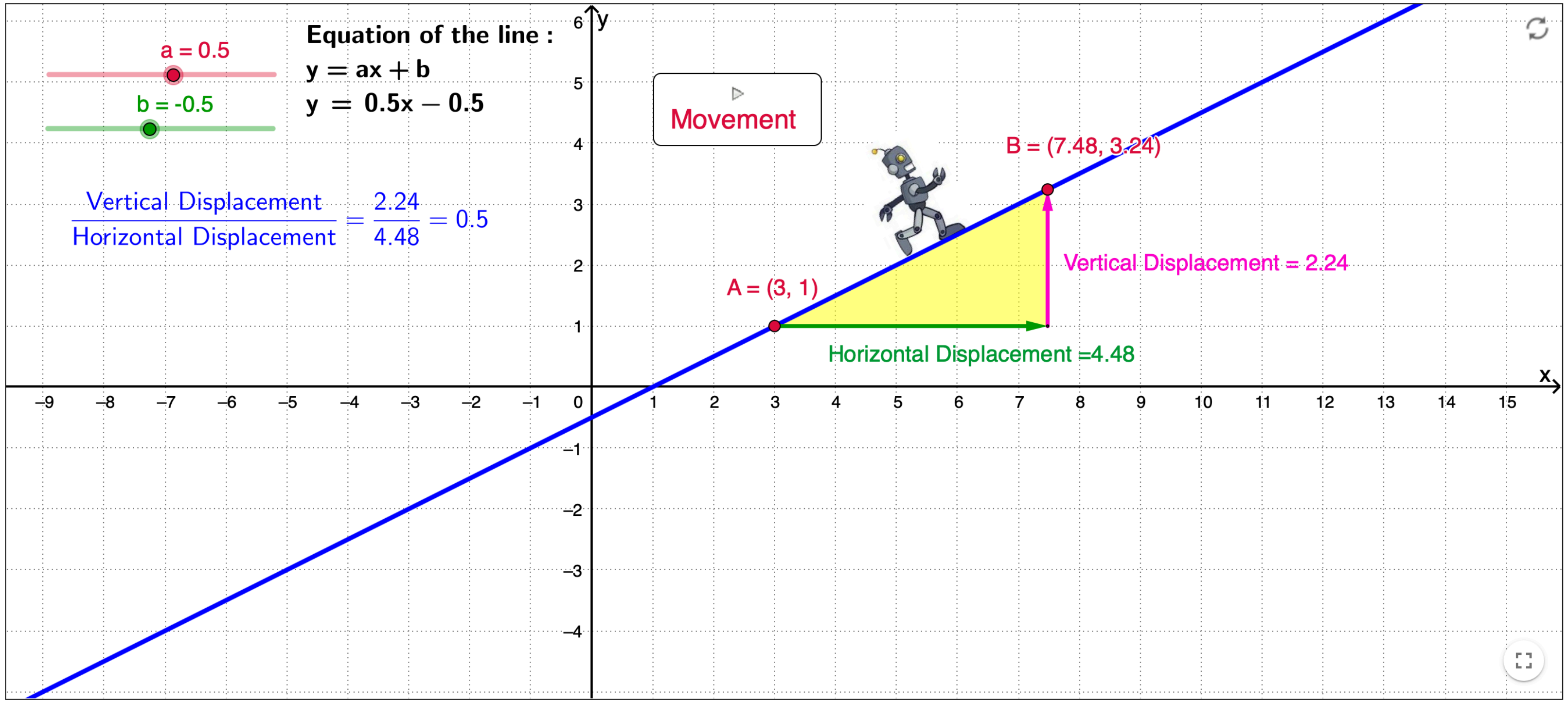
✓ If the robot moves from point *A* to two units to the right, how large will the vertical change be? Does this apply to any starting point *A*?

✓ What do you think will be the vertical change if it moves from point *A*, 5 units to the right? Check your answer with the use of the app.

Precision

Inquiry-based learning

Example: The concept of slope (Athanasiou et al., 2016b, p. 65)



a = 0.5

b = -0.5

Equation of the line :

$y = ax + b$

$y = 0.5x - 0.5$

$$\frac{\text{Vertical Displacement}}{\text{Horizontal Displacement}} = \frac{2.24}{4.48} = 0.5$$

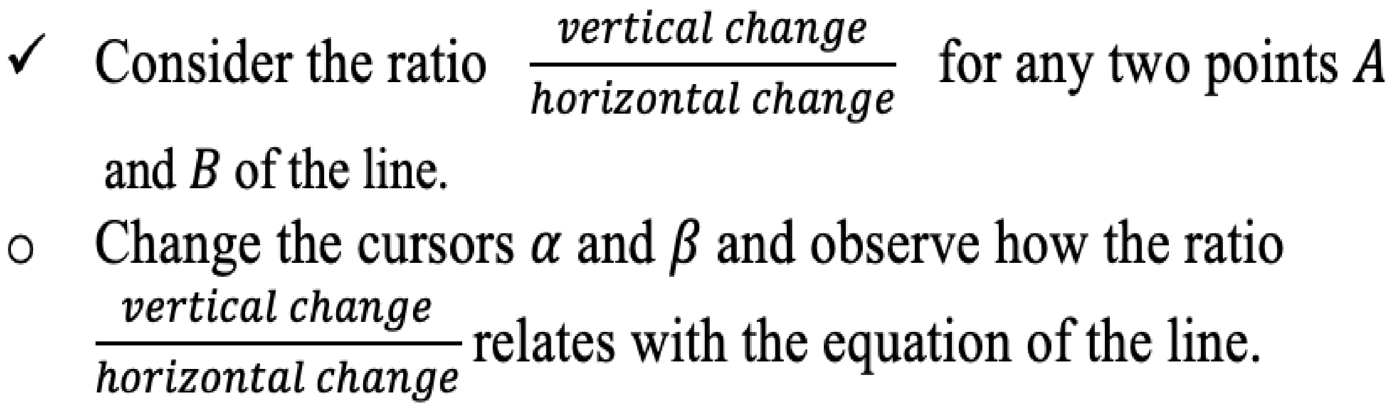
Precision: Explore

Link: https://www.geogebra.org/m/rrqucyde

Inquiry-based learning

Example: The concept of slope (Athanasiou et al., 2016b, p. 65)

Students find a general rule for calculating the slope using two points on the line.



- ✓ Consider the ratio $\frac{vertical\ change}{horizontal\ change}$ for any two points *A* and *B* of the line.
- ○ Change the cursors α and β and observe how the ratio $\frac{vertical\ change}{horizontal\ change}$ relates with the equation of the line.

Generalization

Link: https://www.geogebra.org/m/rrqucyde

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Thank you for your attention and until next time!

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Teacher course: embodied design for functional thinking

Authors / Speakers

Conference, Place

Date

Plan

1. **Hands-on experience with embodied design**
2. **Input on nomograms and embodied design**
3. **Hands-on experience with embodied design nomogram tasks**
4. **Reflecting on the experience with the learning environment, with a focus on embodied design principles**

Hands-on experience with embodied design

Actvity 1

Time: 20 minutes

Material: touch-screen device (phone, tablet, laptop)

Go to website:

https://embodieddesign.sites.uu.nl/activity/

Explore the tasks:

- **Move the points (or one point) and make the feedback green.**
- **Find another green position.**
- **Keep the feedback green (move the points in a way that the feedback would be green all the time). It is rather challenging but possible in each case!**
- **When you are fluent (let your "body" practice at first!), reflect what is the rule that determines the green colour of the feedback.**
- **Move to the next task by listing the tasks on the bottom of the screen.**

Take notes:
- what do you experience when you perform the task?
- a didactical perspective: how does the task invite to mathematical learning?
- what are the common characteristics of this design genre?

Plan

1. **Hands-on experience with embodied design**
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Embodied design

- **The theory of embodied cognition states that cognition cannot be separated from the perception and motor systems in the body**
- **Cognition is fundamentally intertwined with and dependent on action and perception**
- **Perceptual and motor systems solve coordination problem that allow a mathematical perspective on daily base.**
- **Research shows that perceptual and motor systems are involved in cognition on more abstract matters as well**

(**Abrahamson, 2009; Barsalou, 1999; 2008; Shapiro & Stolz, 2019**)

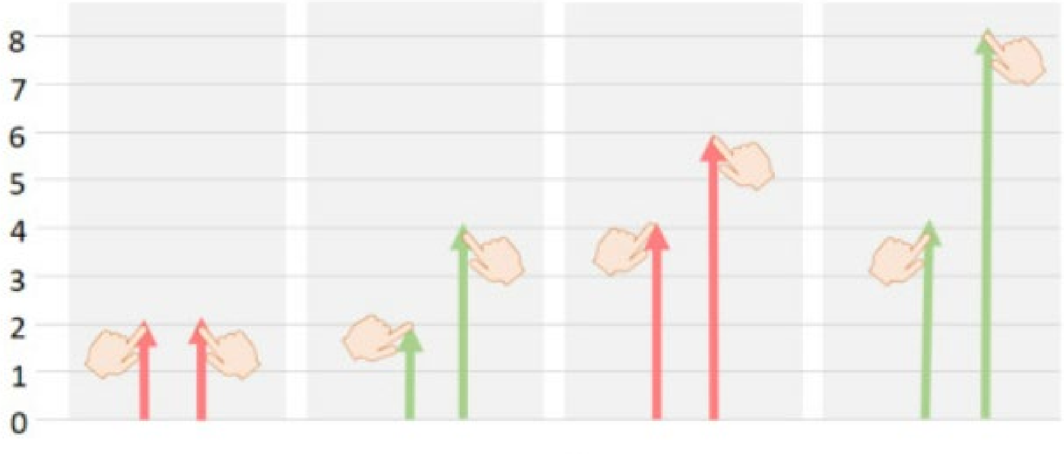
Embodied design

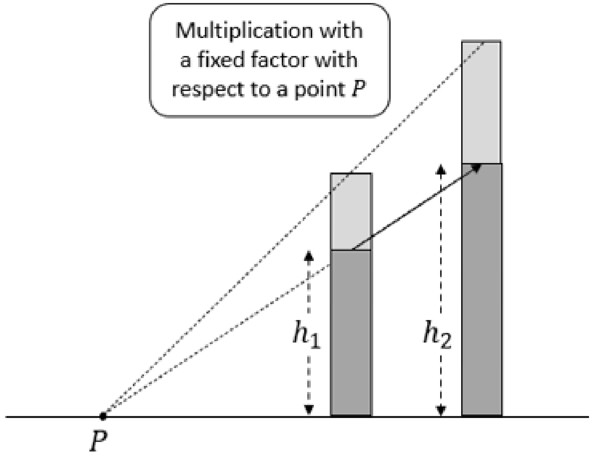
Embodied design in three steps

A motor problem. Learn to move in new ways: a new coordination.

Find a rule that underlies the new movement

Reflect and mathematize the rule/movement





(Alberto et al., 2021)

Nomograms

Duration: 10 minutes

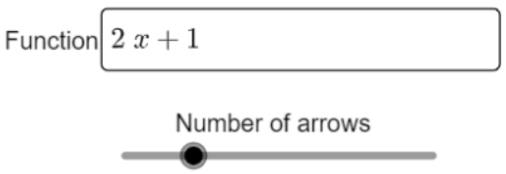
In pairs, explore the module:

Nomogram Intro

Link:
https://app.dwo.nl/embod/?responsive=true&locale=en&profile=108&hash=%23s%3A706362#s:706362

Nomograms

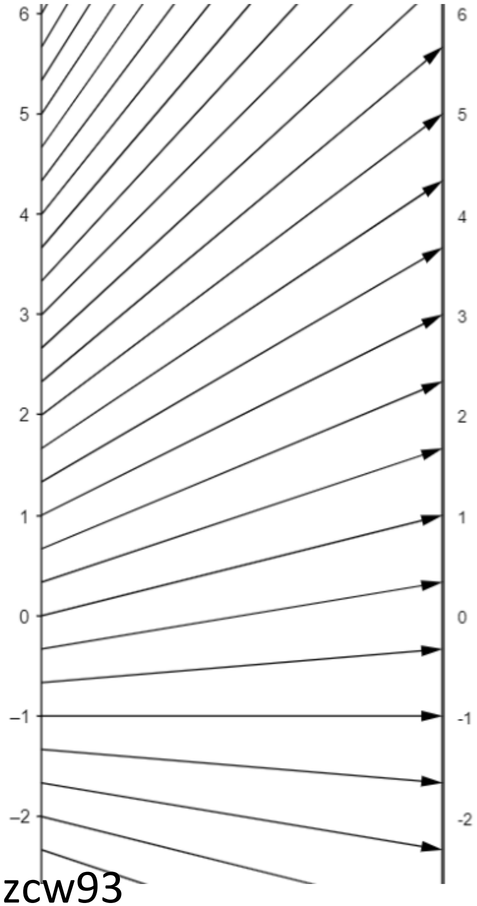
Geogebra
Nomogram app



Function $2x+1$

Number of arrows

Arrows from input values to output values

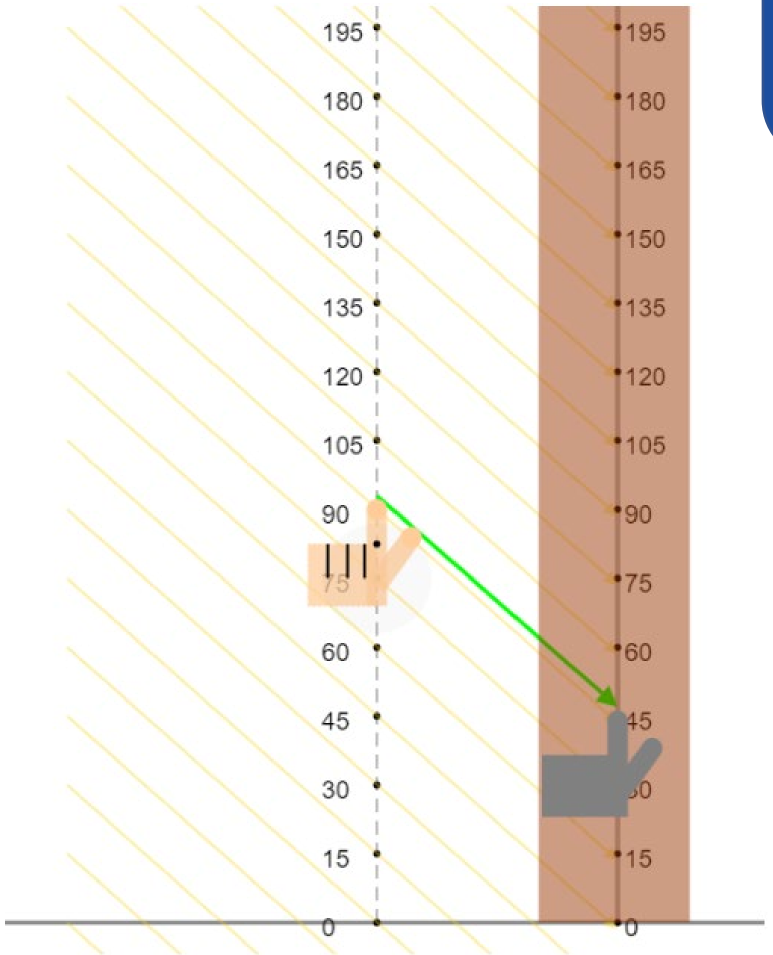


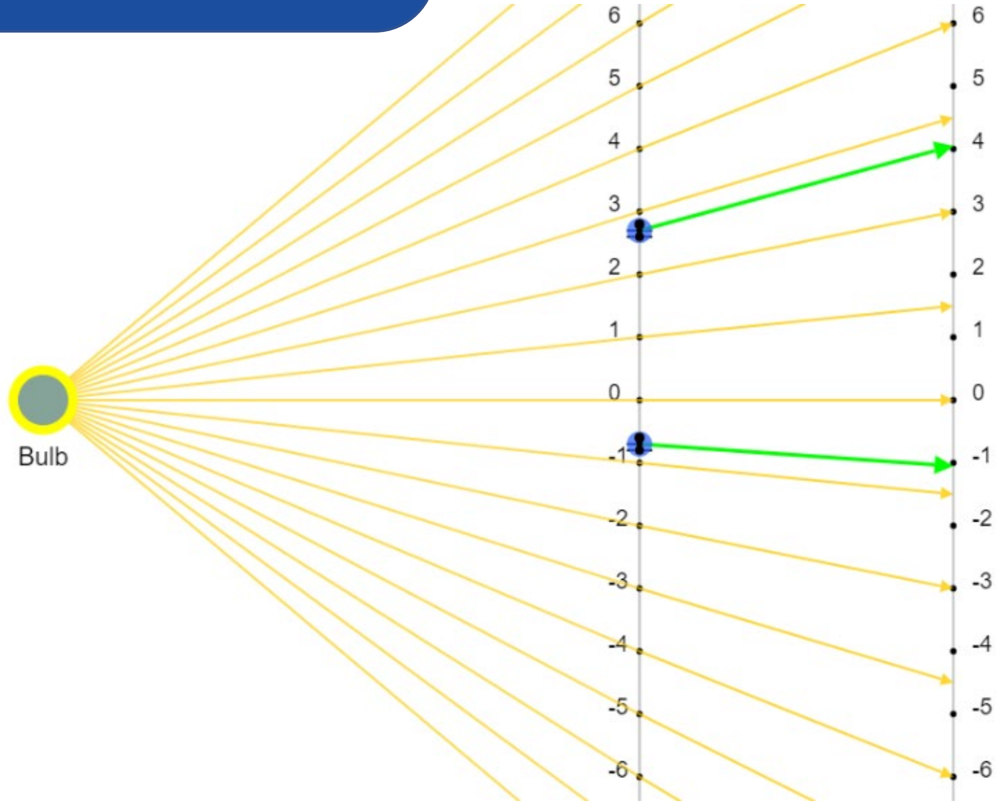
- Emphasizing the input-output character
- An image of the meaning of inverse functions. How?
- An image of composition of functions. How?
- An image of what it means to be additive, linear and non-additive, or constant How?
- An image for domain and range: interval where arrows depart resp. arrive

GeoGebra: https://www.geogebra.org/m/gh7zcw93

Nomograms

Light rays provide a context for a meaningful emergence of the nomogram concept.
We draw a finite number of arrows, but there are an infinite number.





Plan

1. **Hands-on experience with embodied design**
2. **Input on nomograms and embodied design**
3. **<u>Hands-on experience with embodied design nomogram tasks</u>**
4. **Reflecting on the experience with the learning environment, with a focus on embodied design principles**

Embodied design for nomograms

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Duration: 20 minutes

In pairs, explore the modules:

Embodied Design Tasks Nomogram

https://app.dwo.nl/embod/?responsive=true&locale=en&profile=108&hash=%23s%3A706405#s:706405

Embodied Design Tasks Nomogram-Graph relation

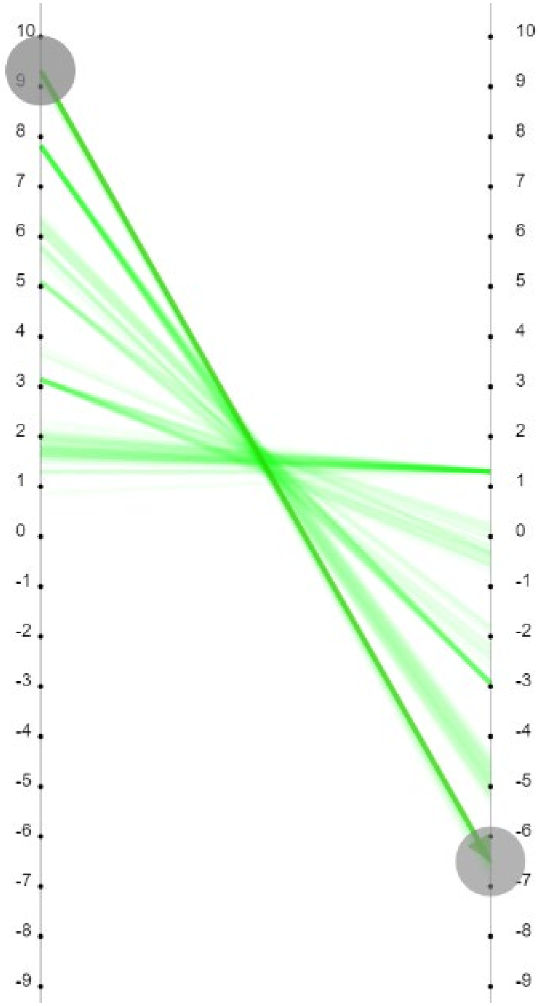
https://app.dwo.nl/embod/?responsive=true&locale=en&profile=108&hash=%23s%3A706770#s:706770

Take notes on three levels on the large sheet:

- **what do you experience themselves when you perform the tasks?**
- **how might the task relate to mathematical learning?**
- **what are the common characteristics of this design genre?**

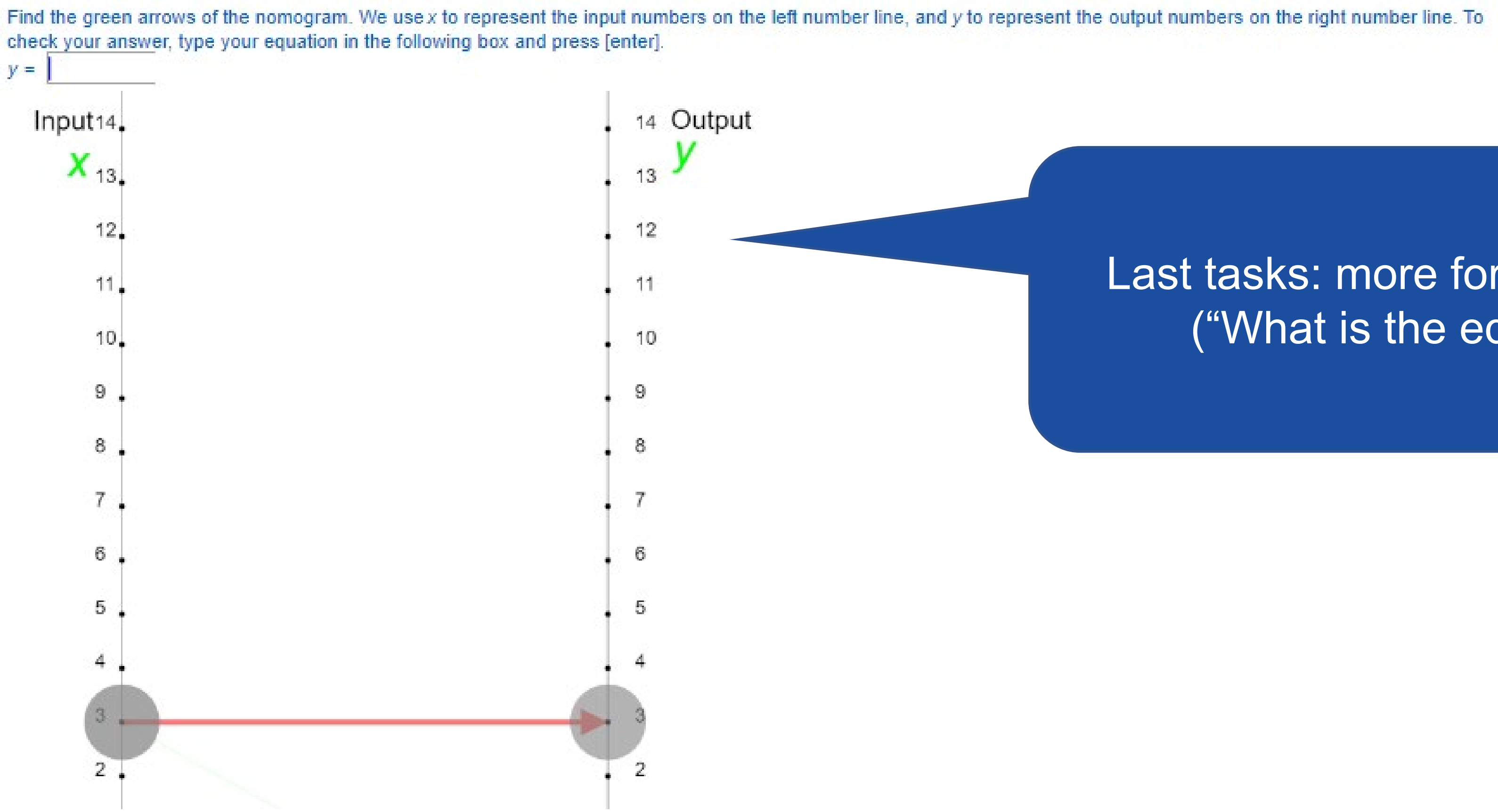
Writes at least three comments/notes.

Embodied design for nomograms



Task 1 – 4 no number lines: focus on movement (“how do you do it?”). Then number lines: mathematization (“what is the rule?”)

Embodied design for nomograms

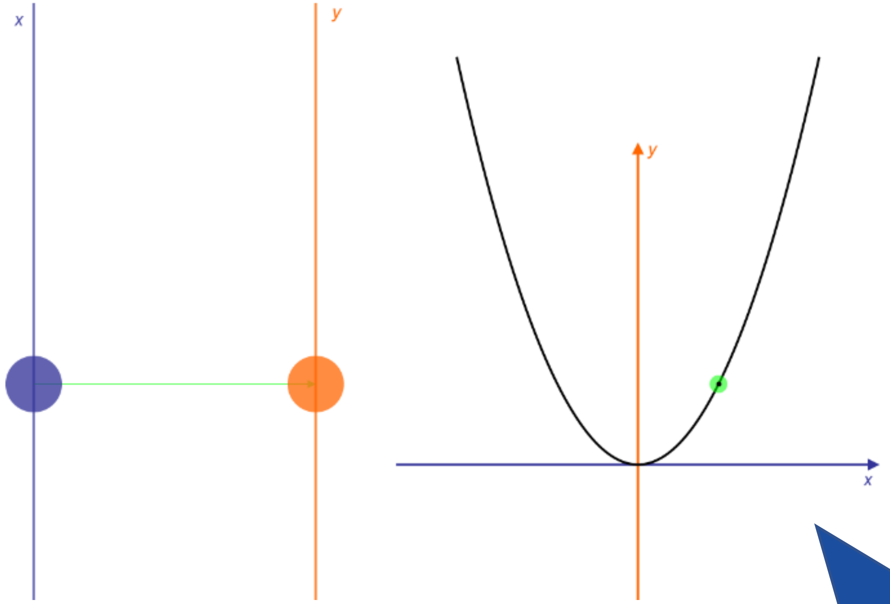


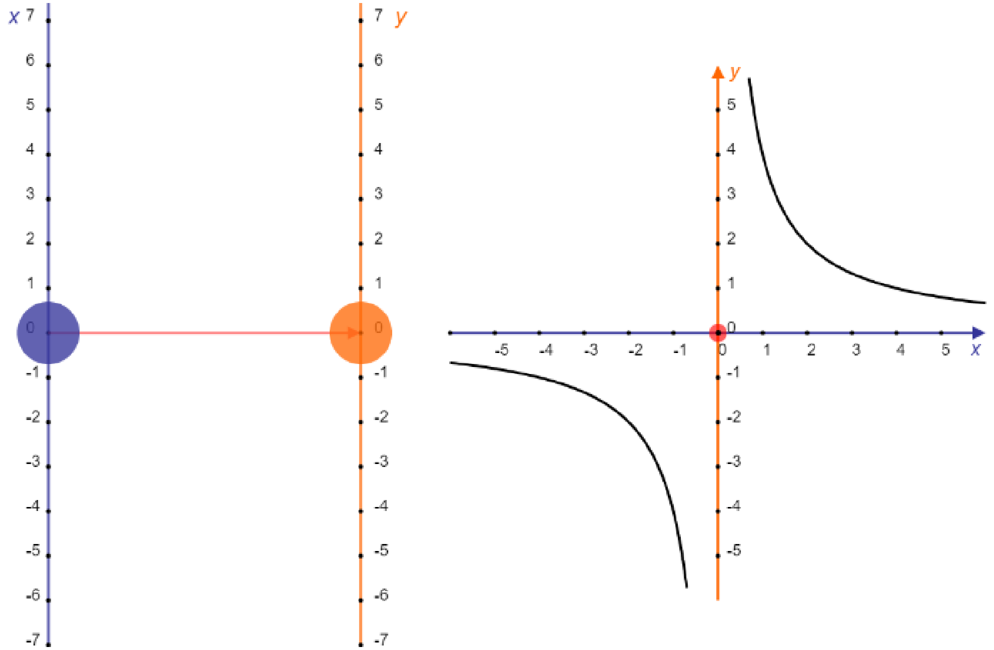
Find the green arrows of the nomogram. We use *x* to represent the input numbers on the left number line, and *y* to represent the output numbers on the right number line. To check your answer, type your equation in the following box and press [enter].

$y =$

Last tasks: more formal/algebraic
(“What is the equation?”)

Embodied design for nomograms





Connect nomogram and graph representation by a form of covariation

Same build up: first without then with numbers

Plan

1. **Hands-on experience with embodied design**
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Embodied design for nomograms

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Duration: 15 minutes

Final discussion

- **Why has the designer of the tasks chosen for these designs?**
- **Would these tasks improve students functional thinking?**
- **What is the interplay between embodiment and functional thinking? Does the aspects of embodiment in these tasks foster/support the development of functional thinking?**

Literature

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Distance-time Learning Environment (Walking Graphs) and Digital Tool Use

Authors / Speaker

Version 30.09.2022

Agenda

- **Introduction with hands-on experience: learning environment walking graphs**
- **Reflection on the experience with the learning environment**
- **Input:**
  - Tool use as a design principle
  - Experiments with real and digital tools to foster functional thinking
- **Reflection on the learning environment with a focus on tool use**

Learning goals:

- Familiarization with the learning environment walking graphs
- Familiarization with tool use as a design principle
- Awareness of current research results regarding experiments with real and digital tools to foster functional thinking

Introduction learning environment: walking graphs

Watch teaser video walking graphs LUE

Introduction learning environment: walking graphs

Task: With help of the provided handouts, experience the learning environment walking graphs.

Keep the following questions in mind:
- What are the learning goals for each task and overall?
- What aspects of functional thinking are addressed?
- What prerequisites are necessary?
- What learning difficulties could appear?
- How do the used tools support the learning process?

Group A: Complete the tasks in the following order:
- Walking graphs physically with the sensor
- Walking graphs physically with the web app: https://tim-lutz.de/funktionenlaufen/indexSelbstZeichnen.html
- Walking graphs digitally with GeoGebra

Group B: Complete the tasks in the following order:
- Walking graphs physically with the web app: https://tim-lutz.de/funktionenlaufen/indexSelbstZeichnen.html
- Walking graphs physically with the sensor
- Walking graphs digitally with GeoGebra

Reflection learning environment: walking graphs

Task: Watch the video. Keep the following questions in mind and discuss in small groups:

- How would you describe the student answers?
- Do you recognize any misconceptions among the students?
- What instruction could be helpful to support the students' functional thinking?
- What else do you notice?

Reflection learning environment: walking graphs

Watch implementation video: walkin graphs zig-zac

Reflection learning environment: walking graphs

Task: Watch the video. Keep the following questions in mind and discuss in small groups:

- How would you describe the student answers for student 1 and 2?
- Do you recognize any misconceptions among the students?
- What instruction could be helpful to support the students’ functional thinking?
- What else do you notice?

Reflection learning environment: walking graphs

Watch implementation video walking graphs exploration

Design principles of the learning environments

INQUIRY-BASED LEARNING

SITUATEDNESS

EMBODIMENT

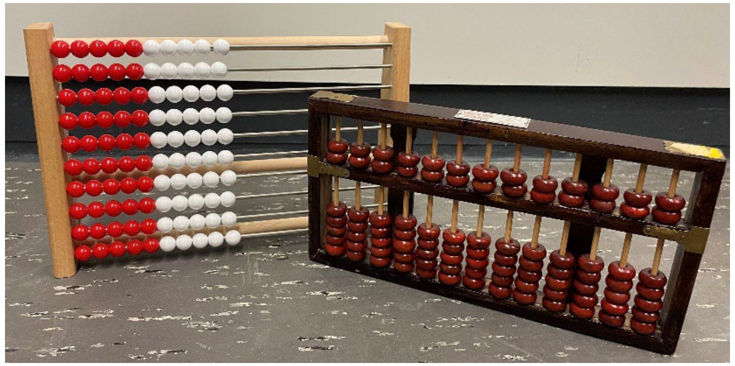
(DIGITAL) TOOLS

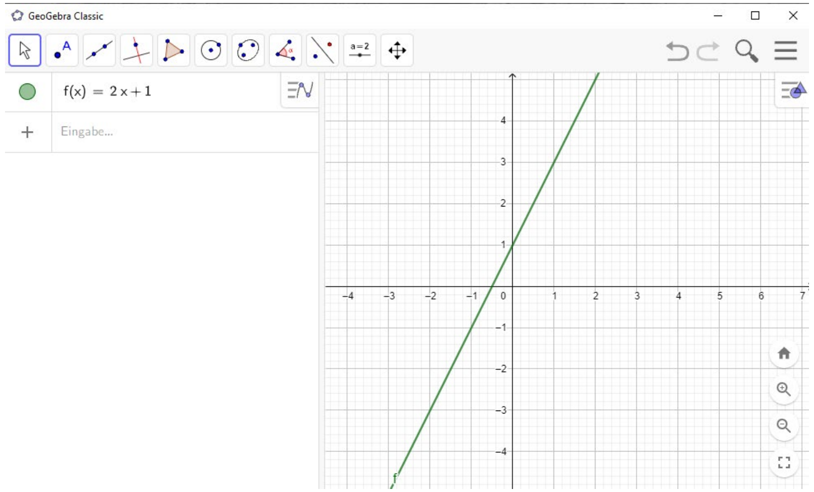
Design principles of the learning environments

(DIGITAL) TOOLS

Tool use: Global description

- **Humans have always been using tools → extend possibilities or simplify tasks**
- **Continuous development of tools → use for cognitive tasks including mathematics** (Monaghan et al., 2016)





$f(x) = 2x + 1$

Tool use: Global description

Using tools for learning mathematics is connected to affordances and constraints

- ➢ **Tools transform mathematical activities (Hoyles, 2018)**
- ➢ **Interplay between tool use and mathematical learning (Drijvers, 2019)**
- ➢ **Much literature available BUT many things still unknown (Drijvers, 2019)**

How to use technology to foster the learning of mathematics?

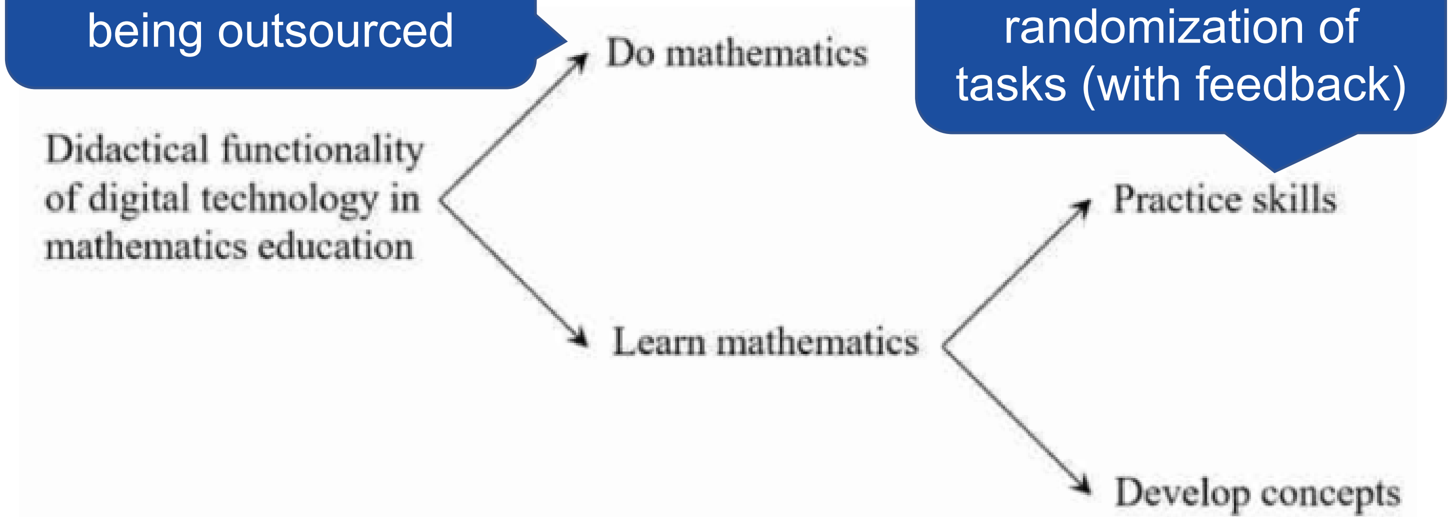
Tool use: dimensions

- **Limited and/ or specific functionality vs. general-purpose tools**
- **Mathematical domains**
- **Didactical functionality**
  - Type of task
  - Teaching and learning process

closely related

Parts of the work are being outsourced

Variation and randomization of tasks (with feedback)



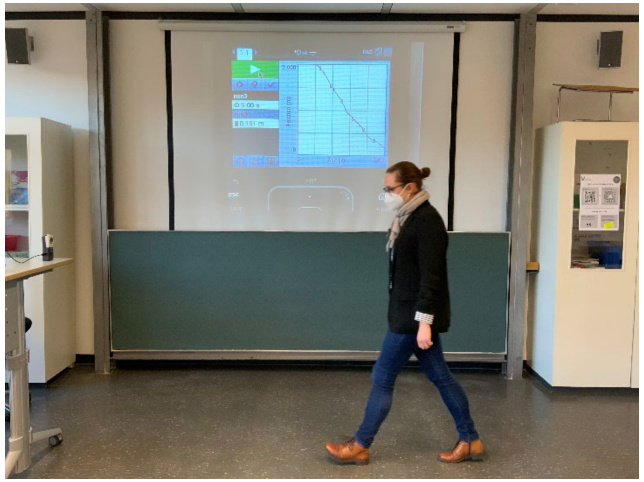
Explore phenomena and conceptual development

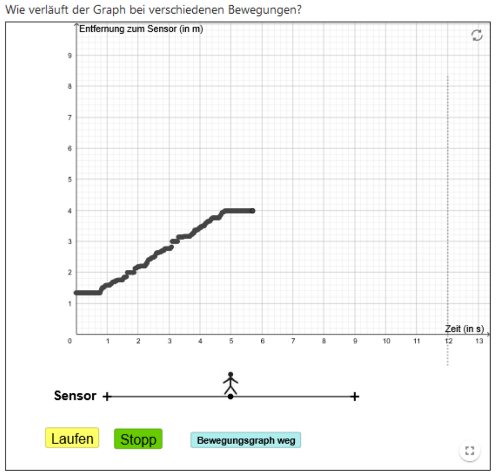
Important for the decision what tool to use & when

Drijvers, 2019, Vision Document FunThink Project

Tool use: dimensions

Task: How do these dimensions (general functionality, mathematical domain, didactical functionality) apply to the tools used in this learning environment?





Wie verläuft der Graph bei verschiedenen Bewegungen?

Sensor and calculator/program

→ sensor & program: limited and specific function, development of concepts

GeoGebra:

→ general: multiple specific functions in different mathematical domains, learning & doing mathematics

→ here: specific functionality in one domain for the development of a concept

Experiments with real and digital tools to foster functional thinking

Using real and digital tools, e.g., to foster functional thinking by experiments

- **Arguments for experiments with real tools:**
  - Functional thinking can be experienced (Ludwig & Oldenburg, 2007)
  - Connection between real world and mathematics (Vom Hofe, 2003)
  - Concept formation (Barzel & Ganter, 2010)
  - Fostering of the correspondence view, reference to real-world; perspective on relationship is rather static (Lichti, 2019)
- **Arguments for experiments with digital tools:**
  - Systematic variation easy to implement (Roth, 2008)
  - Multi-representation-system direcly available (Ballacheff & Kaput, 1997)
  - Learning gains significantly higher than with real tools (Lichti, 2019)
  - Fostering covariation view, reference to graph; perspective on relationship rather dynamical (Lichti, 2019)

Experiments with real and digital tools to foster functional thinking

Using real and digital tools, e.g., to foster functional thinking by experiments

Using real tools: Focus on correspondence

- Value pairs set focus on correspondence (Hoffkamp, 2012)
- Table emphasizes point-by-point reading (Weigand, 1988)
- Static view can hinder covariation (Johnson, 2015)

Using digital tools: Focus on covariation

- Difficult for learners → underdeveloped (Malle, 2000)
- For covariation, local view not sufficient (Leinhardt et al., 1990)
- Qualitative approach to functions (Stellmacher, 1986)

Best effects: Combination of both approaches (Digel et al., 2023)

Experiments with real and digital tools to foster functional thinking

Using real and digital tools, e.g., to foster functional thinking by experiments

Key findings of this study:

- Digital simulations should complement experiments with real materials
  - ➢ Promotion of correspondence view through real experiment
  - ➢ Promotion of covariational view especially through qualitative simulation
  - ➢ Discourse on covariation essential for learning growth
  - ➢ Covariational view accessible even to learners at low skill levels
- Digital learning environments should be embedded in paper-pencil environment
  - ➢ Taking notes / Logging supports reflection
  - ➢ Better availability of the paper-pencil protocol

Best effects: Combination of both approaches
(Digel et al., in press)

Reflection learning environment: walking graphs

Activity: Reflect on the tool use in this module. Exchange you thoughts in small groups.

Summary

What did we learn from today's course?

- **Introduction to the learning environment walking graphs**
- **Hands-on experience with the learning environment**
- **Reflection on the experience with the learning environment with a focus on**
  - Tool use as a design principle
  - Experiments with real and digital tools to foster functional thinking
- **Reflection on the learning environment with a focus on tool use**

Learning goals:

- ➢ Familiarization with the learning environment walking graphs with related learning goals
- ➢ Familiarization with tool use as a design principle
- ➢ Awareness of current research results regarding experiments with real and digital tools to foster functional thinking

Thank you for your attention!

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Forster students‘ functional thinking

Situatedness as design principle

Date

Agenda

- **Situatedness as a design principle**
- **Introduction with hands-on experience: learning environments *number line* and *distance-time (turtle)***

Learning objectives:

- Familiarization with the design principle of situatedness in the context of the modules *number line* and *distance-time (turtle) in* more depth.
- Insights into the support of functional thinking through the learning environments presented.

Situatedness

Origin: Hans Freudenthal (1905-1990) proposes the concept of didactical phenomenology.

Mathematics as a human activity:

> "What humans have to learn is not mathematics as a closed system, but rather as an activity, the process of mathematizing reality and if possible even that of mathematizing mathematics."
>
> (Freudenthal, 1968, p. 7)

Consequence:

Assumption: Knowledge is situated.

- ➢ It manifests itself in everyday activities.
- ➢ It is a result of our actions.
- ➢ It emerges in the context and culture in which it is developed and used.
(Brown, Collins, & Duguid, 1989, p. 32)

Situatedness

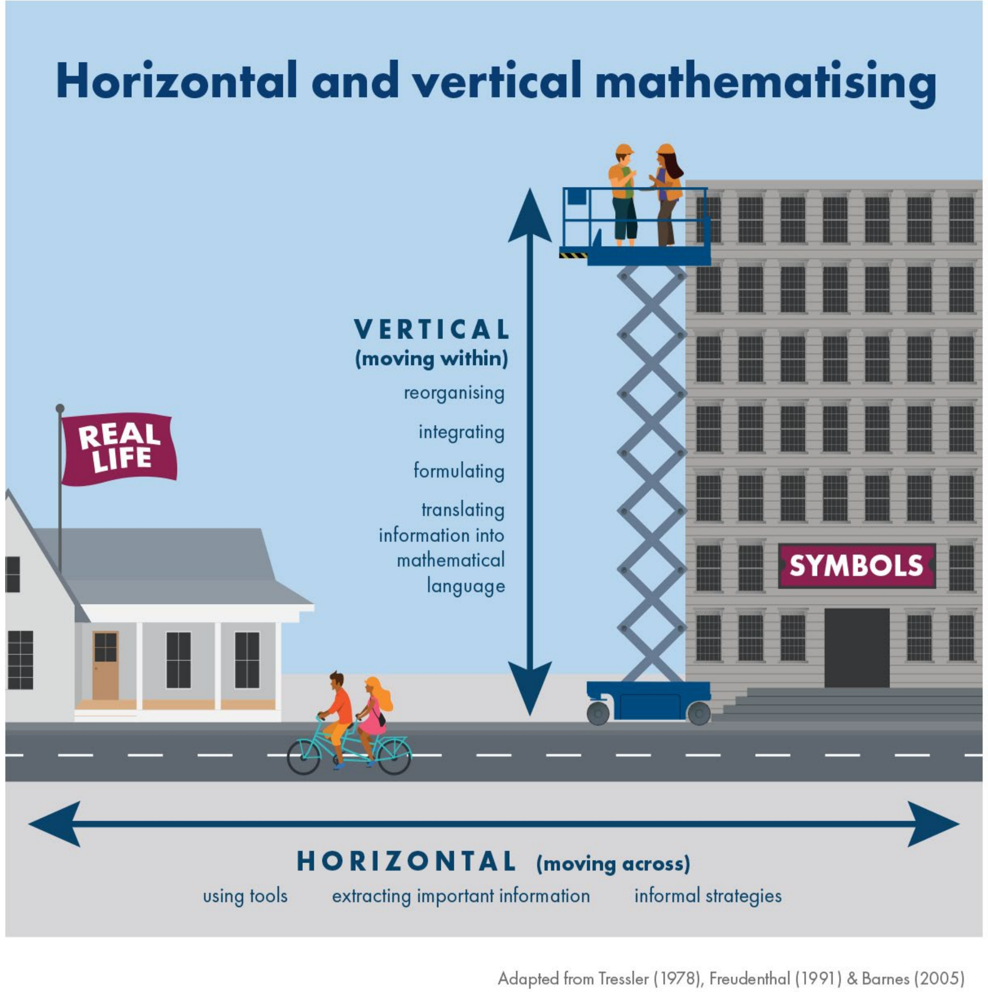


Image: Majewska, D. (2019)

Mathematization

Horizontal mathematization

- "Translation" of a real context into a mathematical model

Vertical mathematization

- "Translation" of a mathematical model into mathematical language - abstract consideration of the model
- Change of representation, solution methods, reorganisation of information, ...

(Van den Heuvel-Panhuizen, M., & Drijvers, P., 2020)

Situatedness

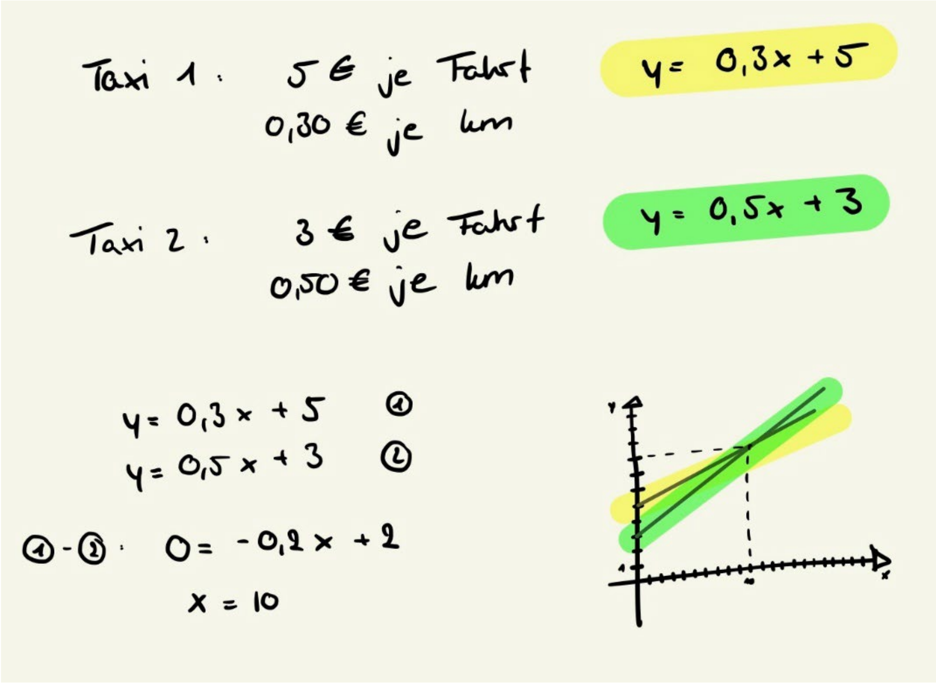
Student task:

Offer Taxi 1:
Each ride costs
5€ and 0,30€
per km.

Offer Taxi 2:
Each ride only €3
and 0,50€ per km.

Compare the two offers. What distance (km) do you have to travel in order that both offers cost the same price?

Student solution:



Taxi 1: 5 € je Fahrt
0,30 € je km

$y = 0,3x + 5$

Taxi 2: 3 € je Fahrt
0,50 € je km

$y = 0,5x + 3$

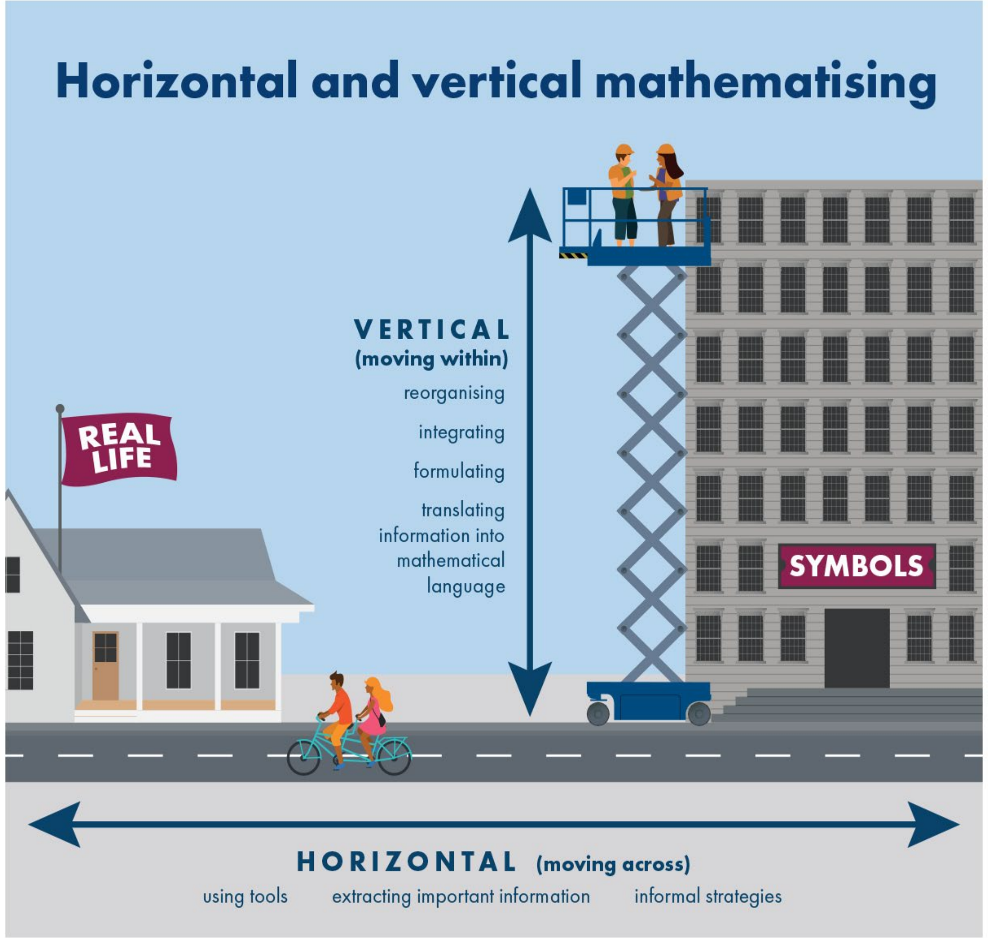
$y = 0,3x + 5$ ①
$y = 0,5x + 3$ ②

① - ②: $0 = -0,2x + 2$
$x = 10$

Activity (5 minutes small group work):
In small groups, identify horizontal and vertical mathematization processes in the given task.

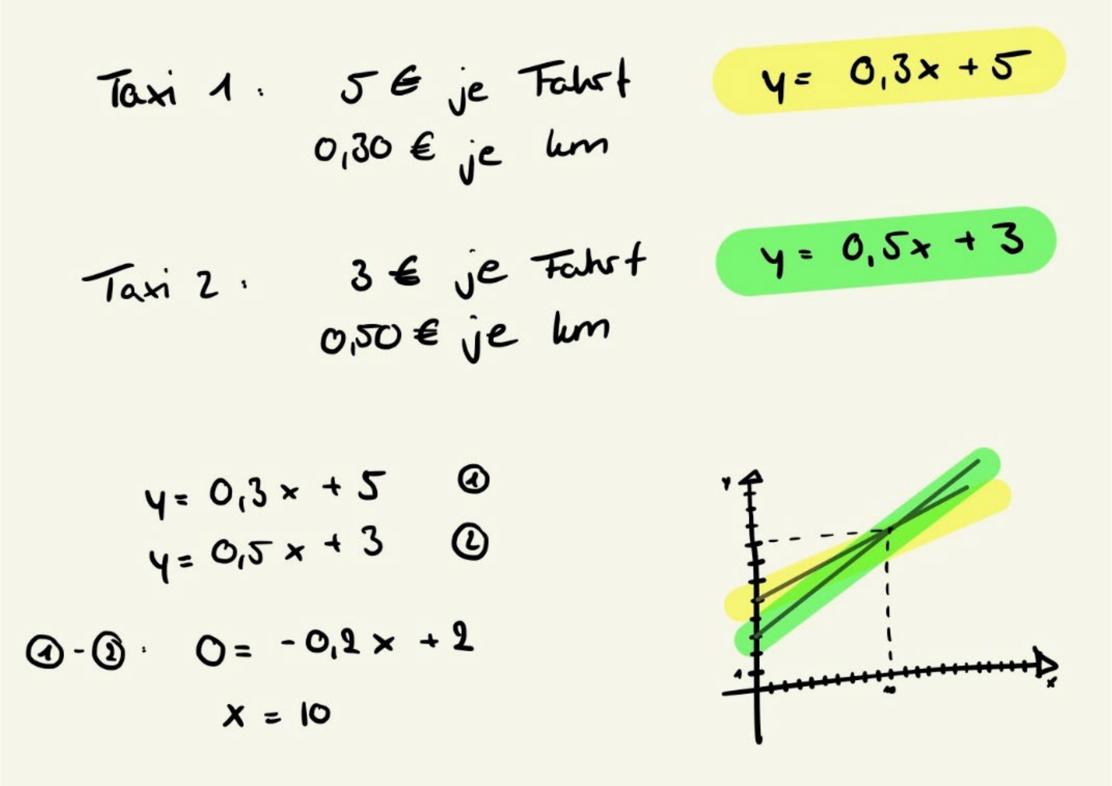
Situatedness

Enhancing functional thinking
from primary to upper secondary school



Adapted from Tressler (1978), Freudenthal (1991) & Barnes (2005)

Majewska, D. (2019)



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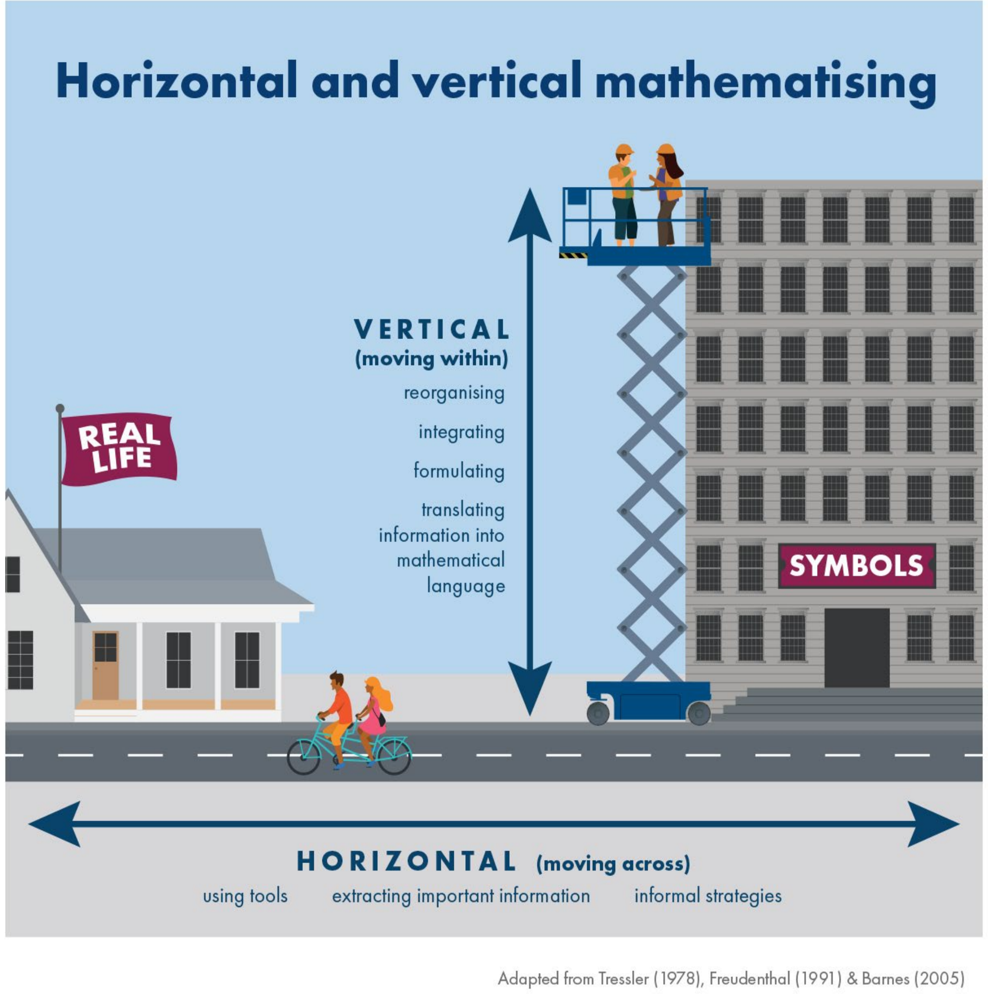
Situatedness



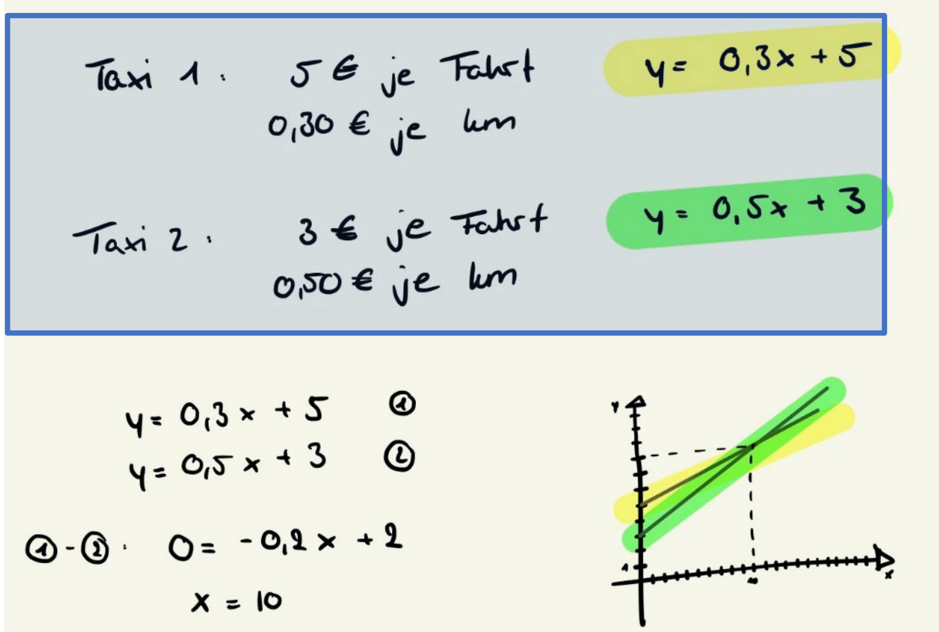
Adapted from Tressler (1978), Freudenthal (1991) & Barnes (2005)

Majewska, D. (2019)

Student task:

What distance (km) do you have to travel in order that both offers cost the same price?

Student solution:



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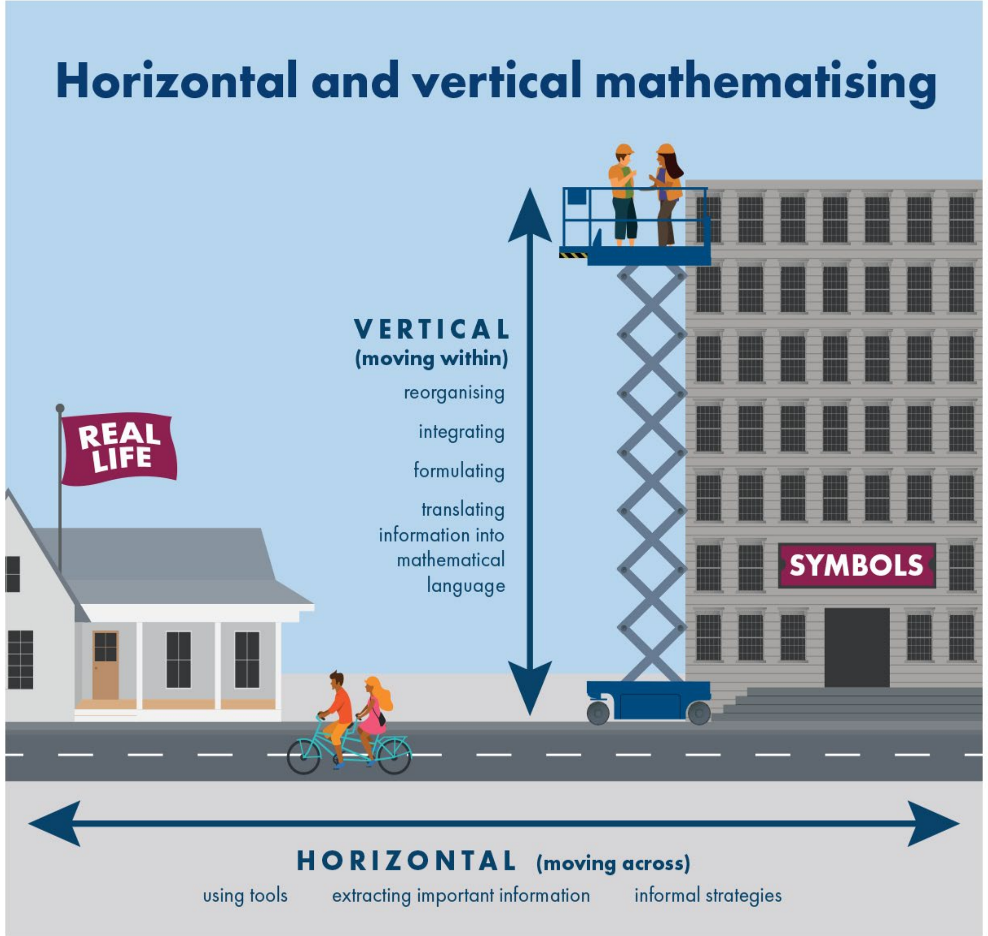
Situatedness



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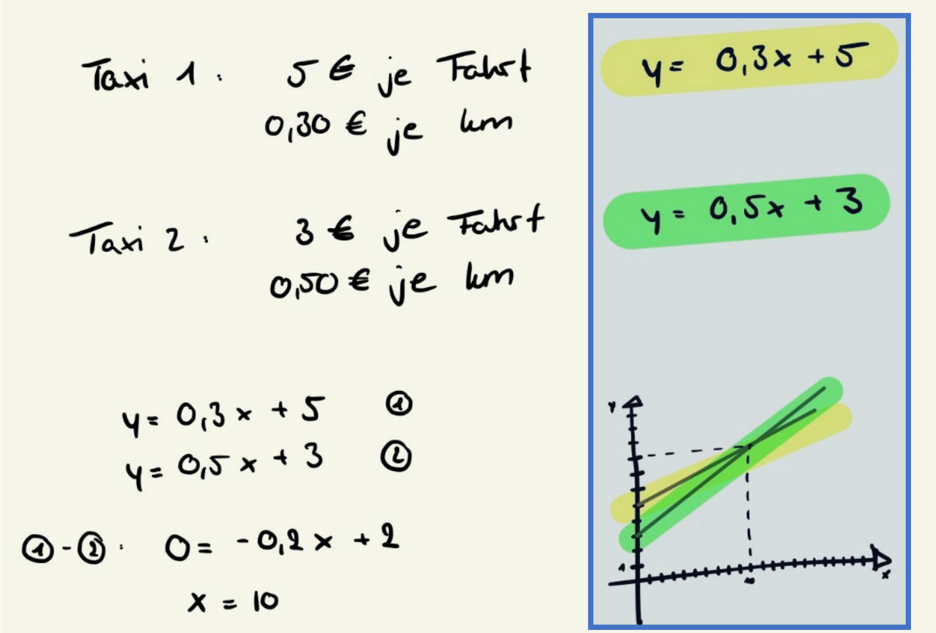

Taxi 1: 5 € je Fahrt
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① - ②: $0 = -0{,}2x + 2$
$x = 10$

$y = 0{,}3x + 5$

$y = 0{,}5x + 3$

Activity (5 minutes small group work):
In small groups, identify horizontal and vertical mathematisation processes in the given task.

Situatedness

Activity (40 minutes group work):

Explore the *number line* and *distance-time (turtle*) learning environments using the provided materials.

In doing so, think about the following questions:

- What are the learning goals of the learning environments and which aspects of functional thinking are addressed?
- What prior knowledge is required?
- What learning difficulties do you anticipate?
- Do mathematization processes take place and if so, which ones?
- To what extent are the activities being worked on situated?

Group A:
Complete the activities in the following order (about 20 min each):

- *Number line*
- *Distance-Time (turtle)*

Group B:
Complete the activities in the following order (about 20 min each):

- *Distance-Time (turtle)*
- *Number line*

Situatedness

Activity:

Explore additional learning environments: *Double Number Line* and *Function Machine*.

In doing so, think about the following questions:

- What are the learning goals of the learning environments and which aspects of functional thinking are addressed?
- What prior knowledge is required?
- What learning difficulties do you anticipate?
- Do mathematization processes take place and if so, which ones?
- To what extent are the activities being worked on situated?

Develop further activities on one of the above learning environments.

Thank you for your attention and until next time!

Co-funded by the Erasmus+ Programme of the European Union

Literature

Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. Educational Researcher, 18(1), 32-42. https://doi.org/10.3102/0013189X018001032

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van den Heuvel-Panhuizen, M., & Drijvers, P. (2020). Realistic mathematics education. In S. Lerman (Ed.), Encyclopedia of Mathematics Education (2nd ed., pp. 713-717). Springer International Publishing. https://doi.org/10.1007/978-3-030-15789-0

Co-funded by the Erasmus+ Programme of the European Union

Misconceptions and learning difficulties

Name of the course

Name of the Lector

Session number:

Intellectual output of FunThink (Erasmus+) project

Brainstorming:
What does come to your mind when you hear:

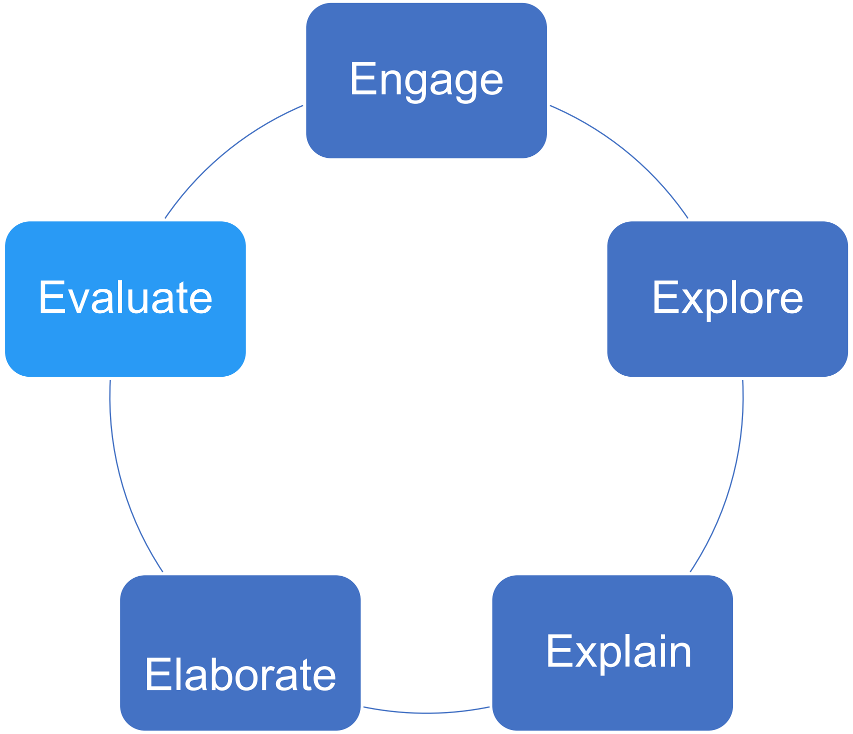
Assessment

Reminder: Inquiry based education

Why is the evaluation inner part of inquiry based education?

Does the teacher always need to assign grades?

Learning cycle 5E



Engage

Explore

Explain

Elaborate

Evaluate

Assessment is communication

- The teacher **gives information** to the student, which is the goal of teaching mathematics.
- The student **informs** how they fulfils these goals.
    - To whom is this information directly accessible?
        - **For the given pupil: an opportunity for self-assessment**
        - **To another student: an opportunity for peer assessment**
        - **To the teacher**

    - What does the teacher do with this information?
        - **He informs the student how he fared compared to the standard / class - classification.**
        - **He informs the student how he can continue to learn, or what progress he has made.**
        - **He adapts teaching.**

Assessment is communication

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Summative

He informs the student how he fared compared to the standard / class - classification.

He informs the student how he can continue to learn, or what progress he has made.

- **He adapts teaching.**

Assessment is communication

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Formative

Components of formative assessment

- Provided by a teacher
- Self-assessment
- Peer-assessment

Types of formative assessment

Proximal FA

"Here and now"

- **Present in verbal communication with pupils**
    - Asking questions
    - Teacher's reaction to the correct answer
    - Teacher's reaction to an incorrect answer
    - The teacher's reaction to the student's initiative
- **Spontaneous, but you can prepare for it**

FA with an instrument

"Pre-prepared tool"

- **Specific formative assessment tools are present**
    - Papers without marks
    - Clear communication of goals (rubrics, checklist, ...)
    - Activity / game
    - Prior assessment of understanding, pre-concepts
    - ...
- **designed in advance**

A good tool for formative assessment in mathematics

- What do I want to formatively assess?
- Is the tool mathematically correct?
- What is its diagnostic potential?

What do I want to formatively assess?

Procedural knowledge

- **Timing:**
  - At the beginning or before starting the unit:
    - To identify level of entry procedural knowledge (e.g. before construction tasks in geometry, identify students level of using geometric tools)
  - During the year:
    - Automation (speed and preciseness) of some procedures (e.g. fluency in using multiplication table)
- **In which case:**
  - Only what makes sense to process automatically (fast and precisely)
  - Only what is already understood

Conceptual knowledge

- **Timing:**
  - At the beginning of the unit:
    - Preconceptions (e.g. before linear function, understand students perceptions of tables)Understanding the input knowledge
    - Understanding of prior knowledge
  - When using inquiry based learning:
    - It can serve as an indirect reference to knowledge which can be used when exploring the topic
  - At the end of the unit:
    - Level of understanding of the concept / solution method / ...
    - Possibility to verify propaedeutic goals as well

Mathematical content of the FA tool

Procedural knowledge

- **Is the tool mathematically correct?**
- **Evaluates processes that need to be automated?**

Conceptual knowledge

- **Is the tool mathematically correct?**

Diagnostic potential of the FA tool

Procedural knowledge

- Does it allow you to evaluate progress? (related to speed or complexity)

Decide whether this is direct proportionality or indirect proportionality.

- Amount of flour for 1 pancake, for *x* pancakes.
- Travel time depending on speed.
- Amount of feed consumed depending on the number of animals.
- Amount of residual feed depending on the number of days.

Conceptual understanding

- Does it allow the teacher to reveal pre-concepts or misconceptions?
- Does it help to identify deep vs. superficial understanding of the concept, method?
- Does it help to identify level of thinking, aspects used?
- Does it inform teacher concerning the next teaching steps?

Today's lesson was:

☹ 😐 🙂

Brainstorming:
What exactly does a teacher do when formatively assessing?

Brainstorming: What exactly does a teacher do when formatively assessing?

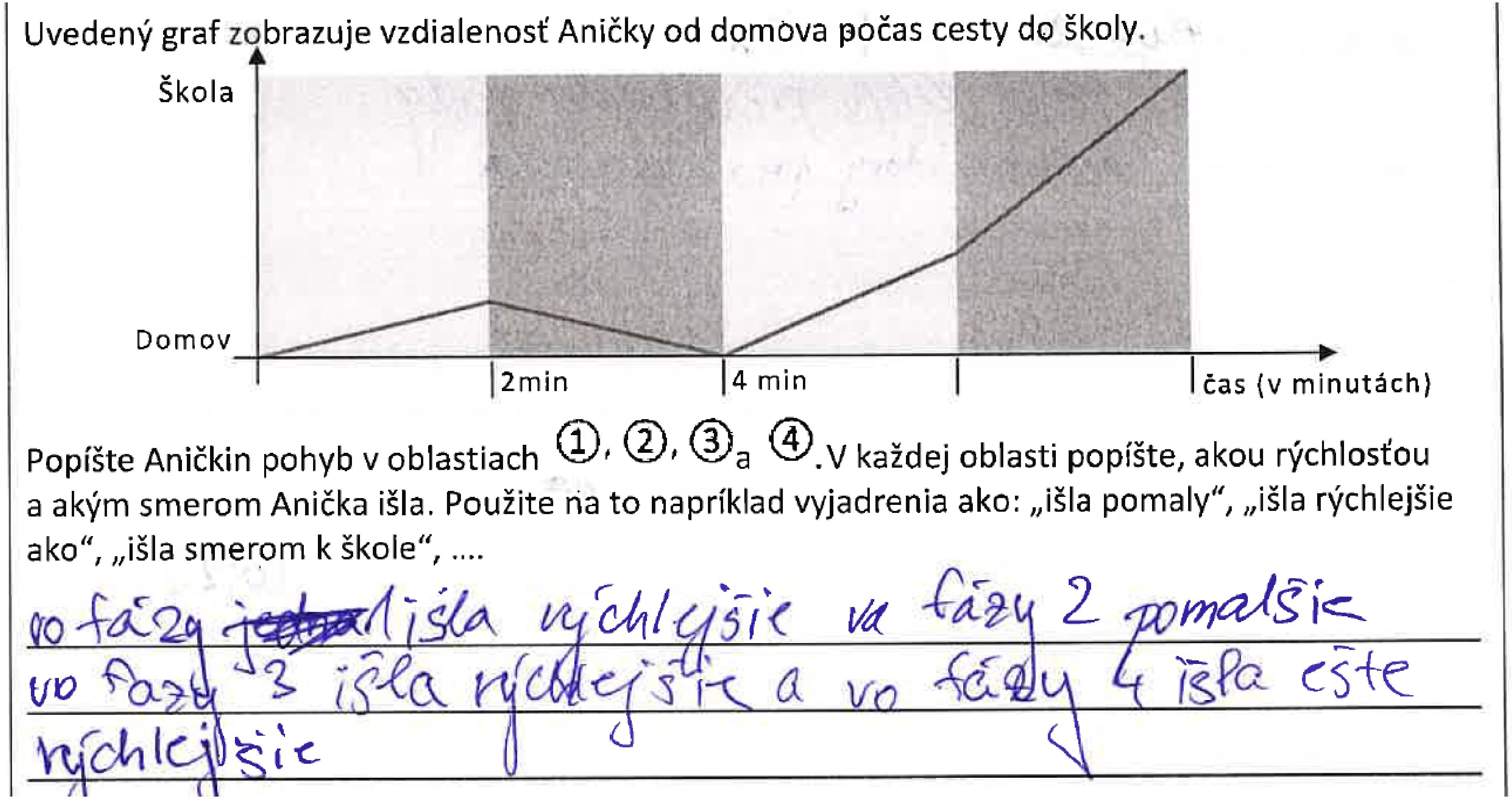
- **Observes and identifies**
    - levels of thinking
    - pre-concepts
    - misconceptions
- **Finds out students' preferences (e.g. which representation is closer to them)**
- **Recognises what language pupils use**
- **Responds to support pupil learning**
    - adapts the language
    - works with pupil error
    - shifts the responsibility for correctness to the pupil
- **Creates pressure for understanding in all pupils**
- ...

- **It is about moving from the role of "know-it-all" to the role of "facilitator"**

Analyse the following student solutions

- **Group work**
  - Analyse the following student solutions. Assess their correctness, identify the representations and aspects of function that the student used.

Solution 1

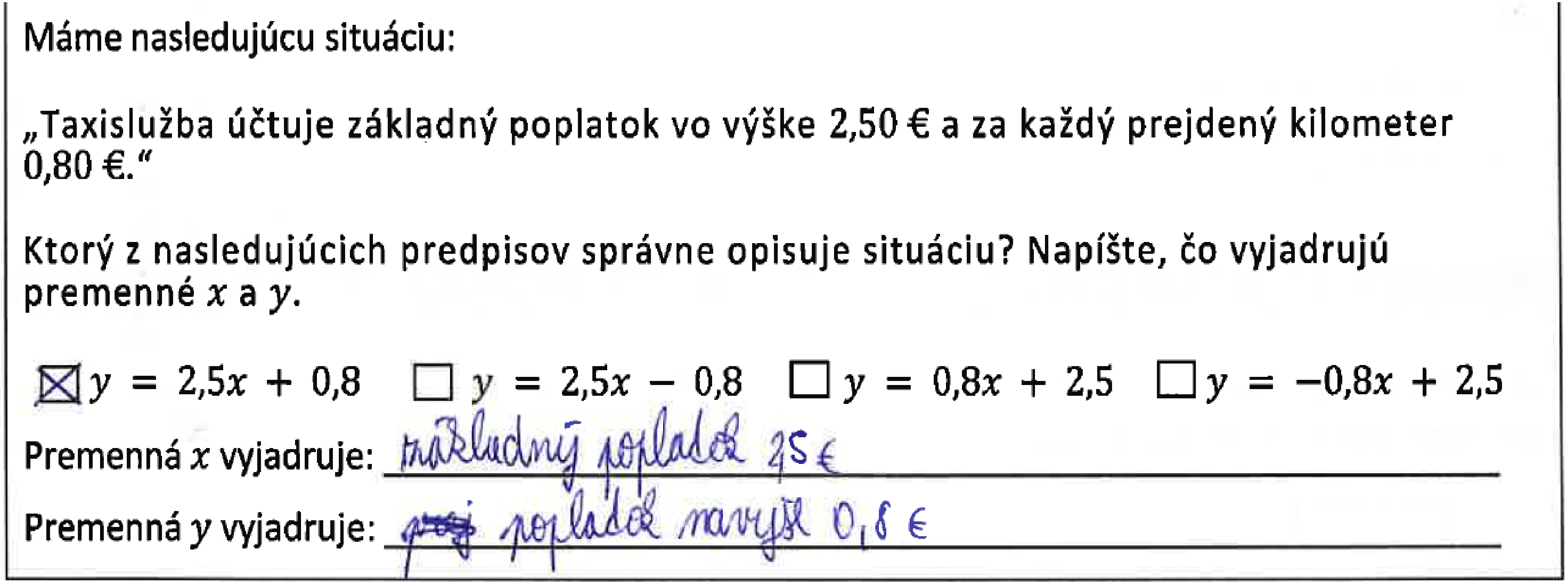


Uvedený graf zobrazuje vzdialenosť Aničky od domova počas cesty do školy.

Popíšte Aničkin pohyb v oblastiach ①, ②, ③ a ④. V každej oblasti popíšte, akou rýchlosťou a akým smerom Anička išla. Použite na to napríklad vyjadrenia ako: „išla pomaly", „išla rýchlejšie ako", „išla smerom k škole", ...

vo fázy ~~jedna~~ išla rýchlejšie na fázy 2 pomalšie
vo fázy 3 išla rýchlejšie a vo fázy 4 išla ešte
rýchlejšie

Solution 2



Máme nasledujúcu situáciu:

„Taxislužba účtuje základný poplatok vo výške 2,50 € a za každý prejdený kilometer 0,80 €.“

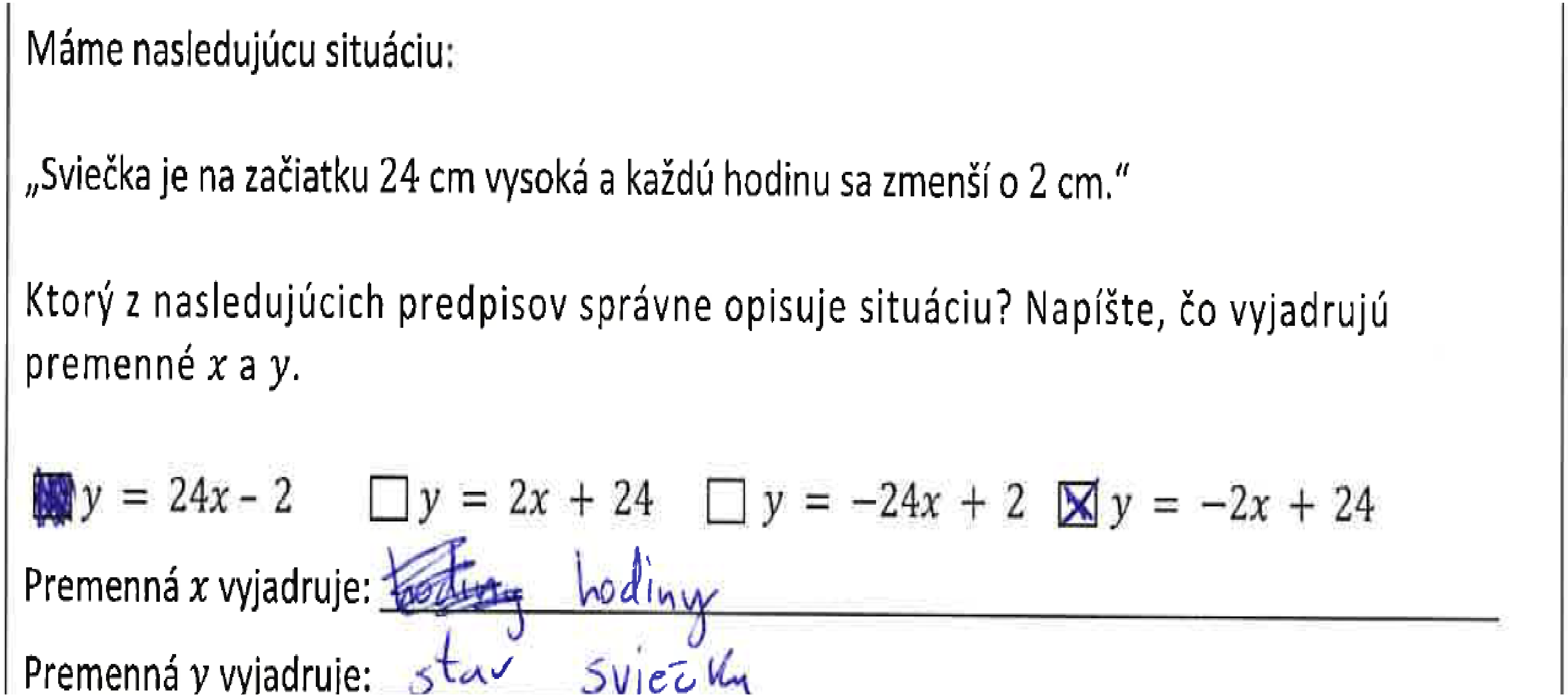
Ktorý z nasledujúcich predpisov správne opisuje situáciu? Napíšte, čo vyjadrujú premenné x a y.

☒ $y = 2{,}5x + 0{,}8$ ☐ $y = 2{,}5x - 0{,}8$ ☐ $y = 0{,}8x + 2{,}5$ ☐ $y = -0{,}8x + 2{,}5$

Premenná x vyjadruje: základný poplatok 2,5 €

Premenná y vyjadruje: ~~prej~~ poplatok navyše 0,8 €

Solution 3



Máme nasledujúcu situáciu:

„Sviečka je na začiatku 24 cm vysoká a každú hodinu sa zmenší o 2 cm.“

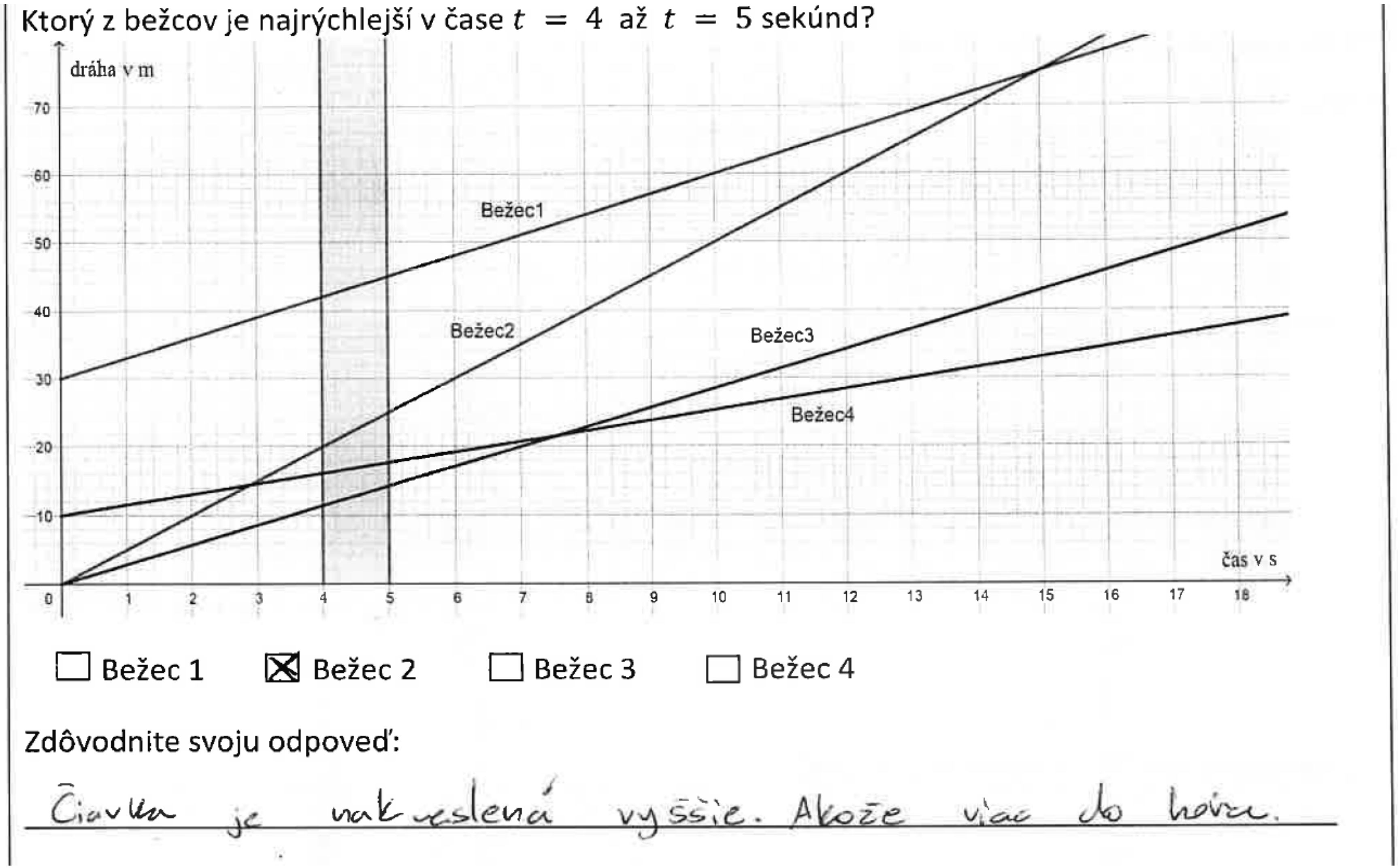
Ktorý z nasledujúcich predpisov správne opisuje situáciu? Napíšte, čo vyjadrujú premenné x a y.

■ $y = 24x - 2$ ☐ $y = 2x + 24$ ☐ $y = -24x + 2$ ☒ $y = -2x + 24$

Premenná x vyjadruje: ~~[illegible]~~ hodiny

Premenná y vyjadruje: stav sviečky

Solution 4



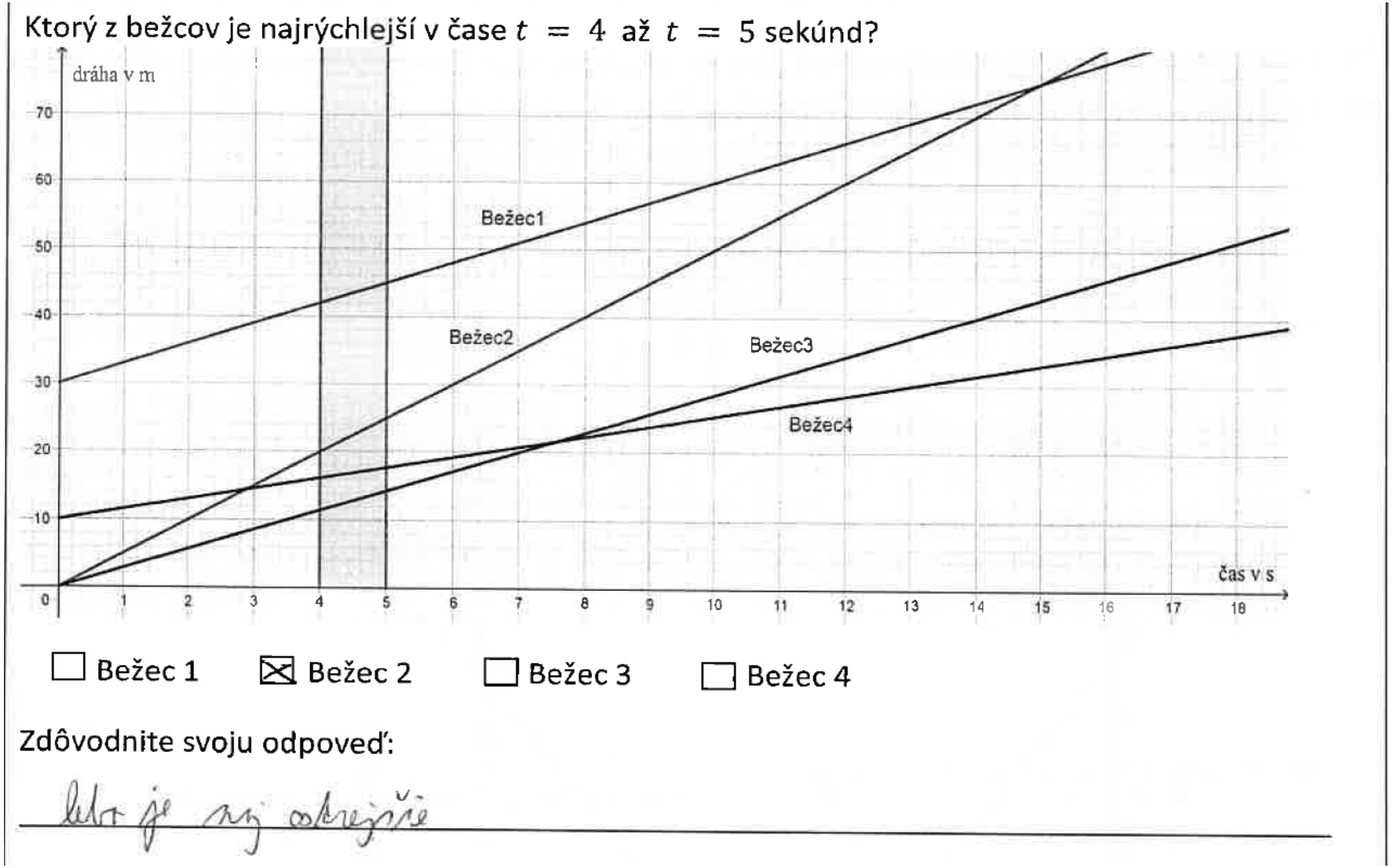
Ktorý z bežcov je najrýchlejší v čase $t = 4$ až $t = 5$ sekúnd?

☐ Bežec 1 ☒ Bežec 2 ☐ Bežec 3 ☐ Bežec 4

Zdôvodnite svoju odpoveď:

Čiarka je nakreslená vyššie. Akože viac do hora.

Solution 5



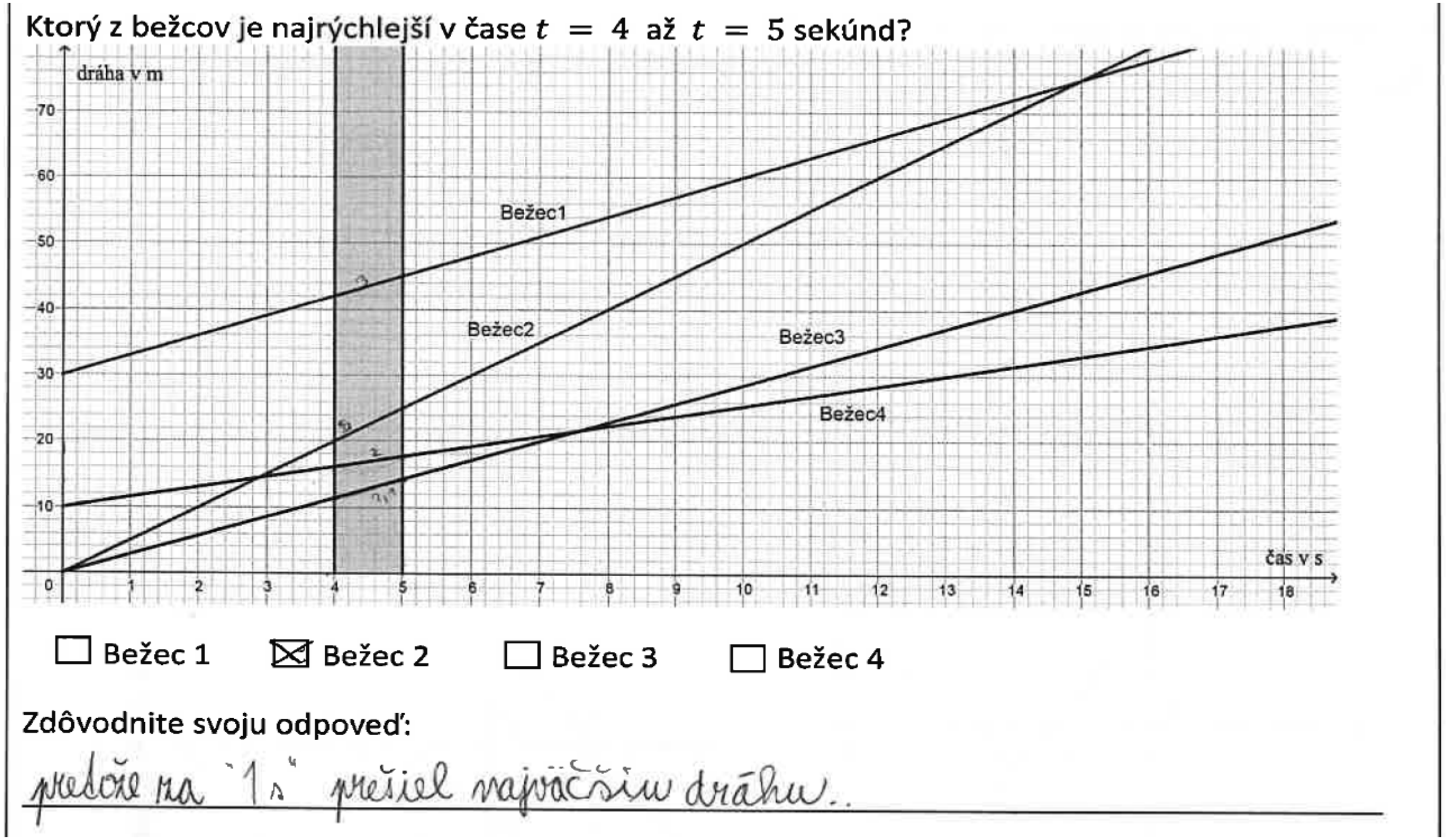
Ktorý z bežcov je najrýchlejší v čase $t = 4$ až $t = 5$ sekúnd?

☐ Bežec 1 ☒ Bežec 2 ☐ Bežec 3 ☐ Bežec 4

Zdôvodnite svoju odpoveď:

lebo je naj ostrejšie

Solution 6



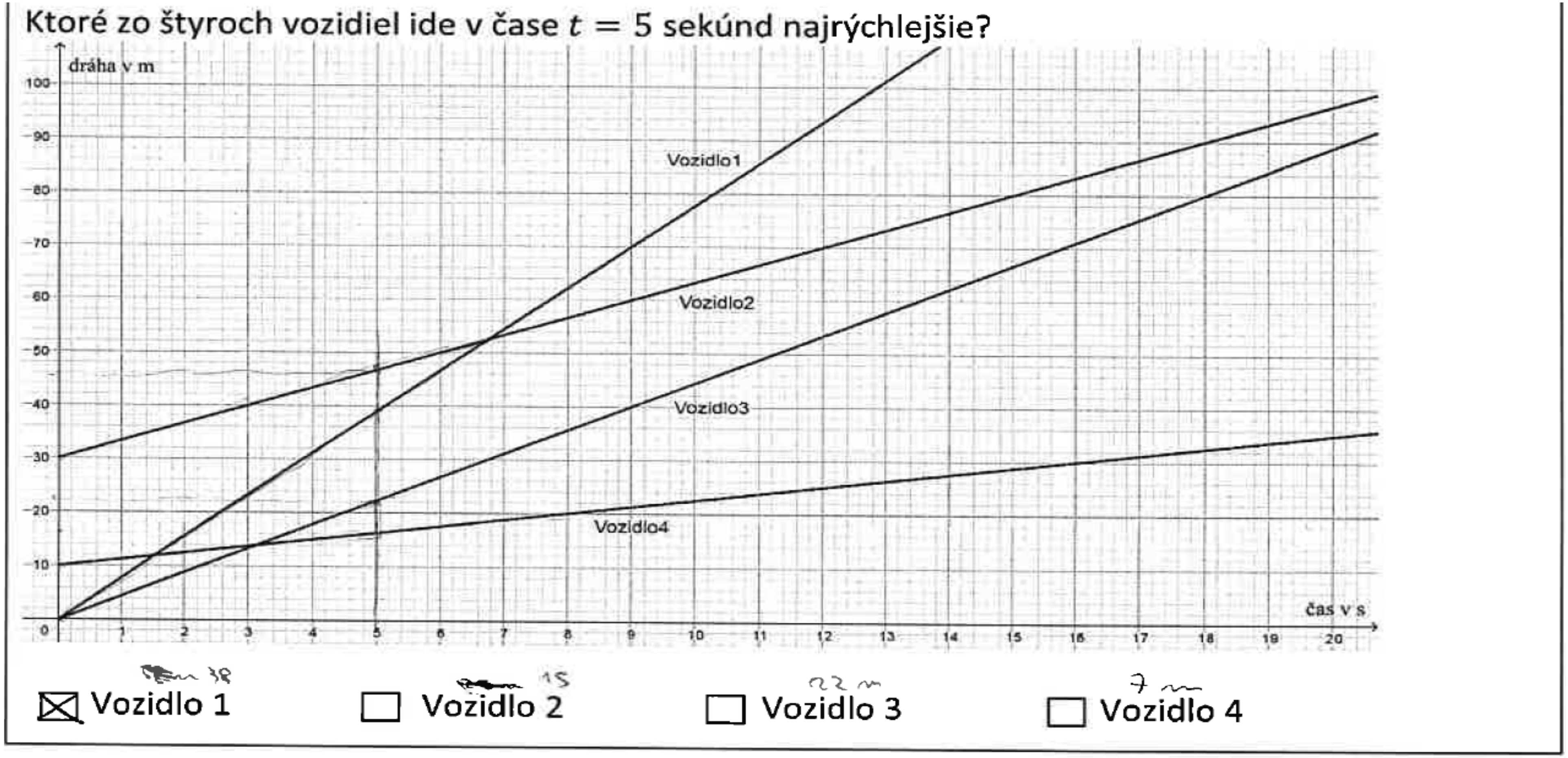
Ktorý z bežcov je najrýchlejší v čase $t = 4$ až $t = 5$ sekúnd?

☐ Bežec 1 ☒ Bežec 2 ☐ Bežec 3 ☐ Bežec 4

Zdôvodnite svoju odpoveď:

pretože na „1s" prešiel najväčšiu dráhu..

Solution 7



Ktoré zo štyroch vozidiel ide v čase $t = 5$ sekúnd najrýchlejšie?

38 15 22 m 7 m

☒ Vozidlo 1 ☐ Vozidlo 2 ☐ Vozidlo 3 ☐ Vozidlo 4

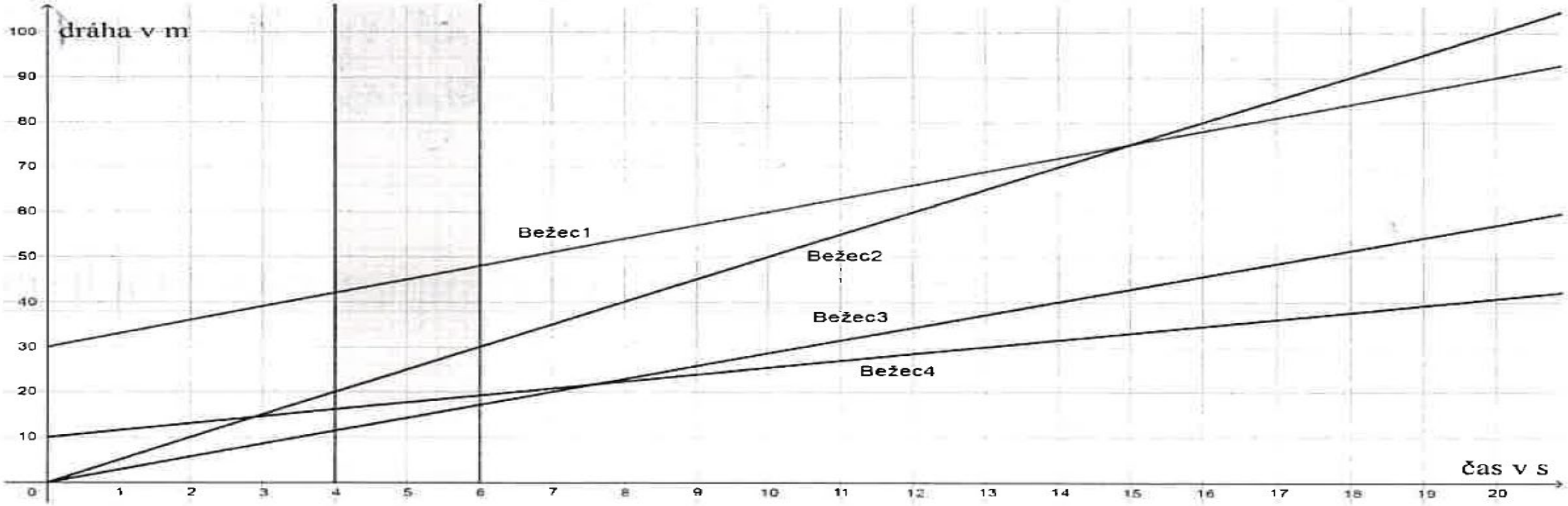
Solution 8

a) Koľko metrov zabehne Bežec 2 v časovom rozmedzí $t = 4$ s až $t = 6$ s?

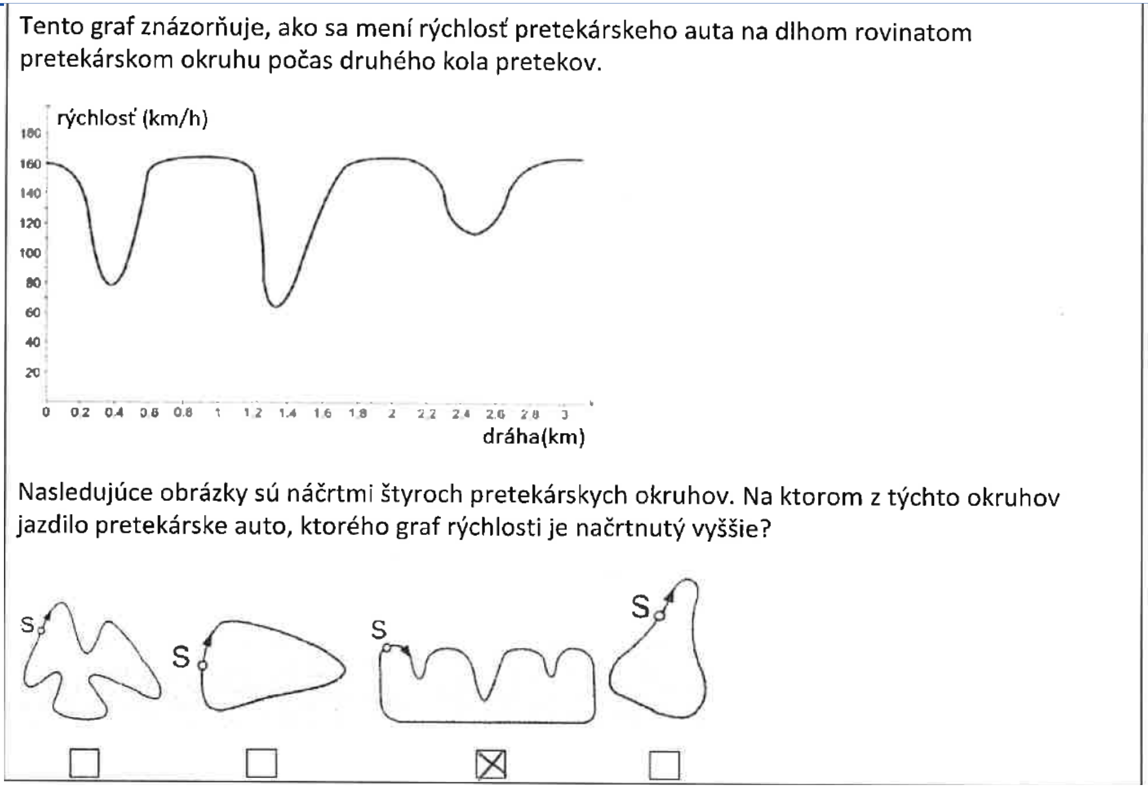
Odpoveď: v čase t=4 bežec 2 zabehne zhruba 25m a v čase t=6 ≐ 30m

b) Kedy je Bežec 1 rýchlejší ako Bežec 2?

Odpoveď: Bežec 1 je rýchlejší od času t=1s až t=15s



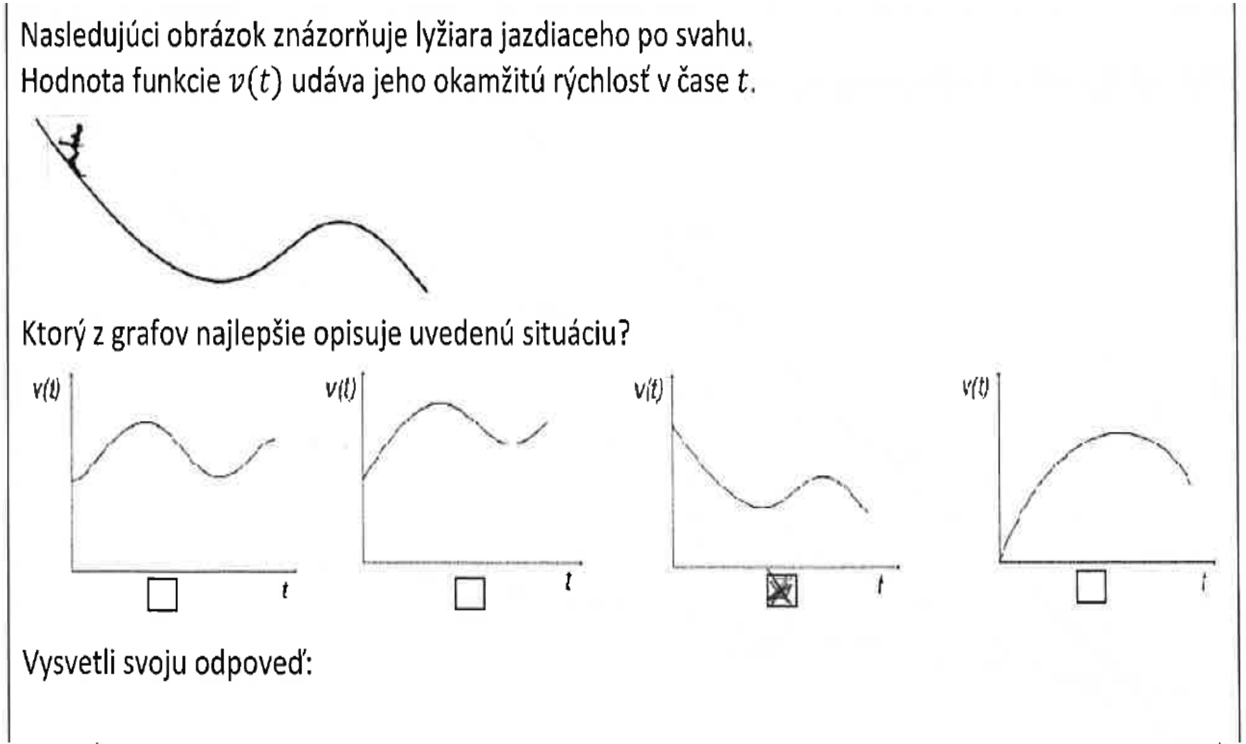
Solutions 9 -12



Tento graf znázorňuje, ako sa mení rýchlosť pretekárskeho auta na dlhom rovinatom pretekárskom okruhu počas druhého kola pretekov.

Nasledujúce obrázky sú náčrtmi štyroch pretekárskych okruhov. Na ktorom z týchto okruhov jazdilo pretekárske auto, ktorého graf rýchlosti je načrtnutý vyššie?

☐ ☐ ☒ ☐

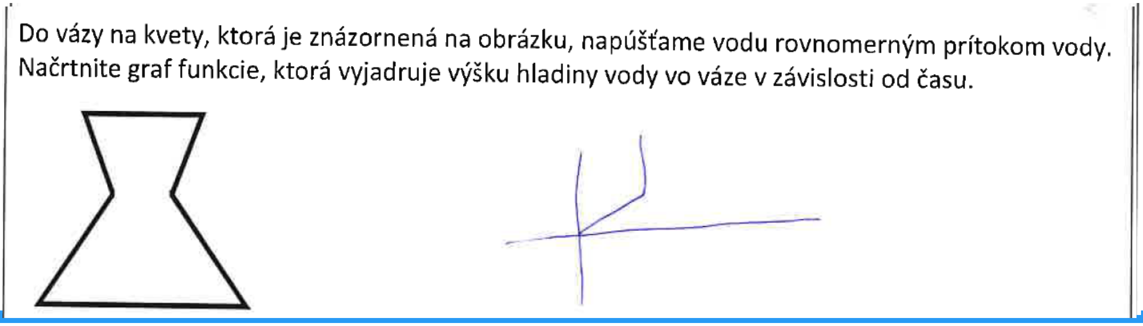


Nasledujúci obrázok znázorňuje lyžiara jazdiaceho po svahu. Hodnota funkcie $v(t)$ udáva jeho okamžitú rýchlosť v čase t.

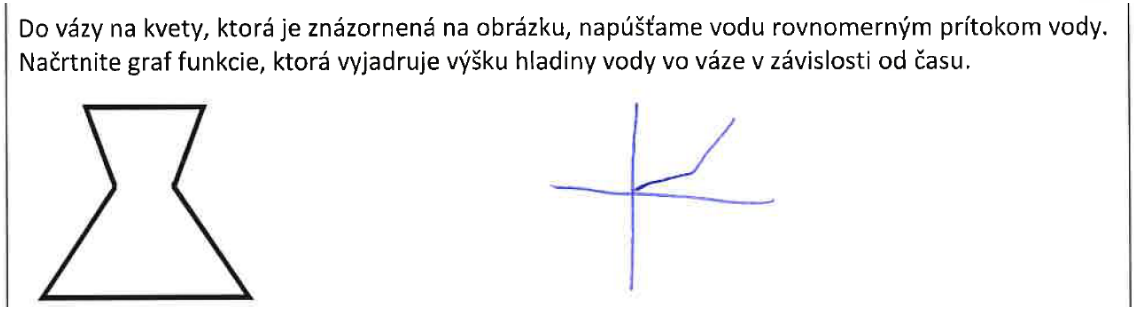
Ktorý z grafov najlepšie opisuje uvedenú situáciu?

☐ ☐ ☒ ☐

Vysvetli svoju odpoveď:



Do vázy na kvety, ktorá je znázornená na obrázku, napúšťame vodu rovnomerným prítokom vody. Načrtnite graf funkcie, ktorá vyjadruje výšku hladiny vody vo váze v závislosti od času.



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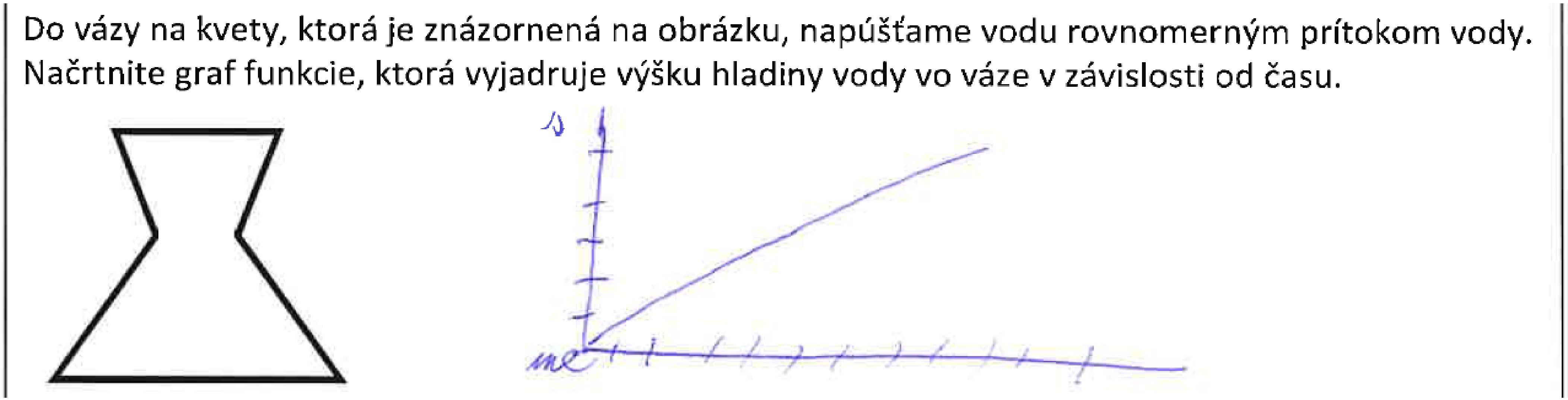
Solution 13



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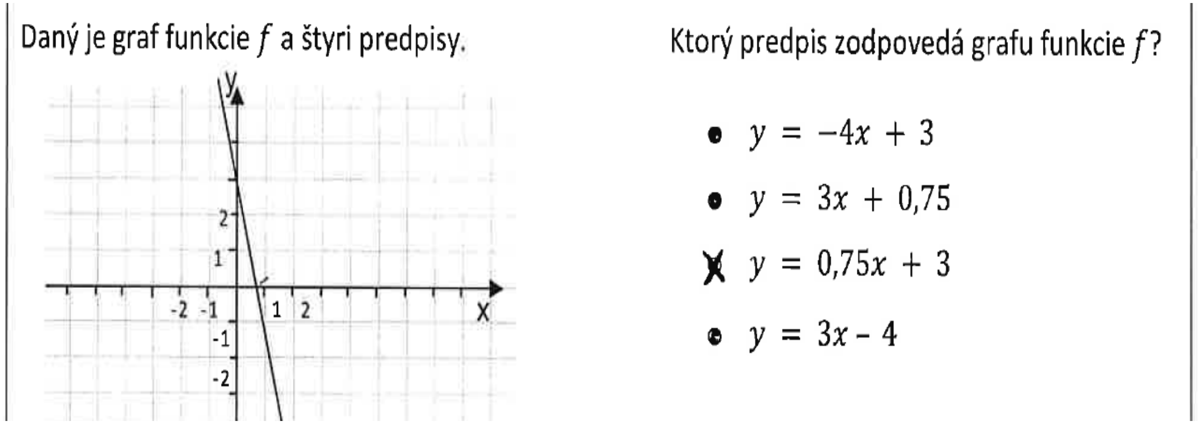
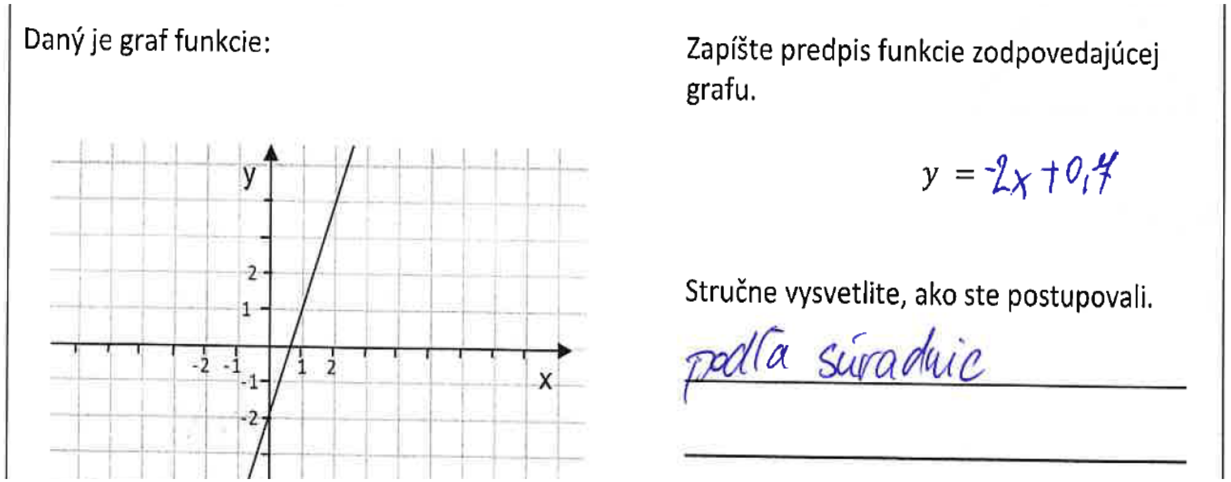
Solutions 14-16



Daný je graf funkcie f a štyri predpisy.

Ktorý predpis zodpovedá grafu funkcie f?

- ● $y = -4x + 3$
- ● $y = 3x + 0{,}75$
- ☒ $y = 0{,}75x + 3$
- ● $y = 3x - 4$



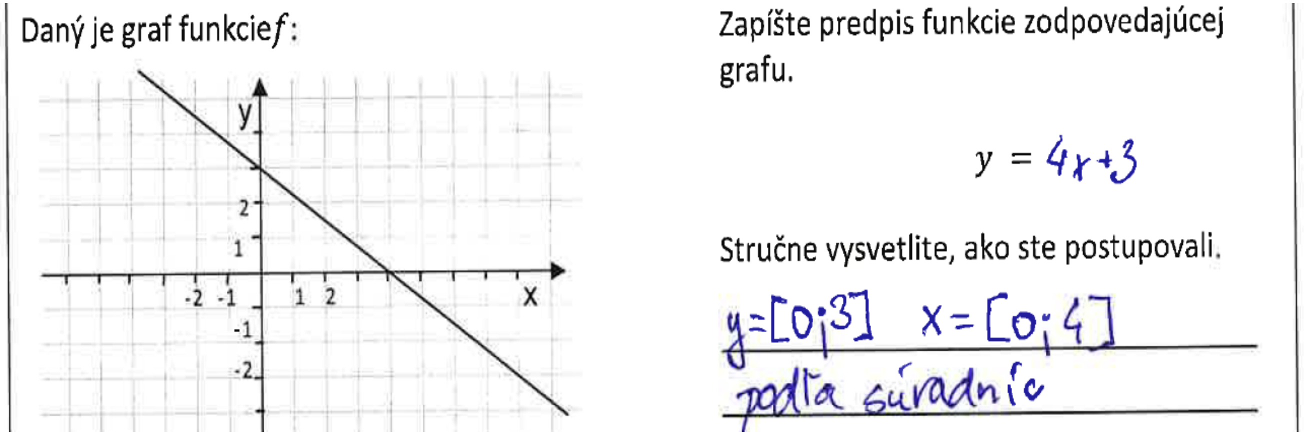
Daný je graf funkcie:

Zapíšte predpis funkcie zodpovedajúcej grafu.

$y = -2x + 0{,}4$

Stručne vysvetlite, ako ste postupovali.

podľa súradnic



Daný je graf funkcie f:

Zapíšte predpis funkcie zodpovedajúcej grafu.

$y = 4x+3$

Stručne vysvetlite, ako ste postupovali.

y=[0;3] x=[0;4]

podľa súradníc

Brainstorming:
What misconceptions might students have about the topic of function?

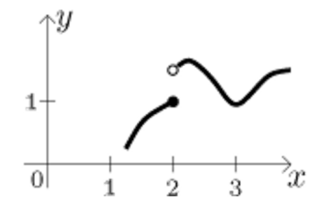
- **Error vs. Misconception**
  - Misconception - a misconception about some concept, property
  - Error - incorrect solution, answer for any reason - inattention, different understanding of the task, misconception
- **In small groups, write as long a list as possible of misconceptions that we might encounter specifically when teaching functions.**

Misconceptions, errors and difficulties in teaching functions

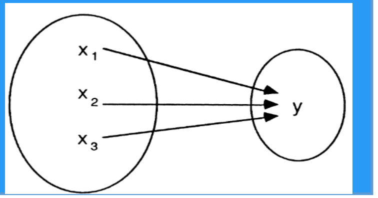
1. What is function?

- inaccurate ideas about what graphs of functions should look like (only graphs that show an obvious or simple pattern are understood as graphs of functions
- the idea that only graphs where a pattern is apparent represent functions; others look strange, artificial or unnatural.
- the ideas that the following functions are not functions:functions composed of arbitrary correspondences, functions given by more than one rule, and functions that are not officially recognized and labeled as functions by mathematicians
- the idea that functions must consist of quantities which are variable
- the idea that a function implies causality
- the belief that functions are always simple, the chaos between assigning "one to many" and "many to one"

Example:
The graph does not represent a function because it does not exhibit an obvious pattern, a regularity.



Example:
This is not a function because several x's point to one y.

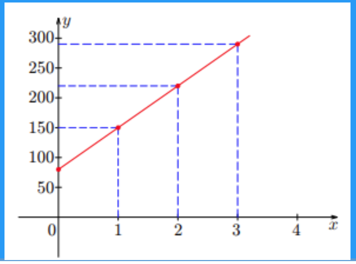


Misconceptions, errors and difficulties in teaching functions

2. Linearity

- the tendency to define a function as a relationship that, when represented graphically, produces a linear pattern
- the tendency to connect every two consecutive points with a straight line (in both contextual and abstract situations)
- only one function can pass through two given points (a generalization of a special property of linear functions)
- over-generalisation of properties of linear functions to other types of functions

Example:
Alex bought a new car with an odometer reading 80 km. However, that is about to change as he is going on a bigger trip tomorrow. Describe how (table, graph, equation) if his average speed is 70 km per hour.

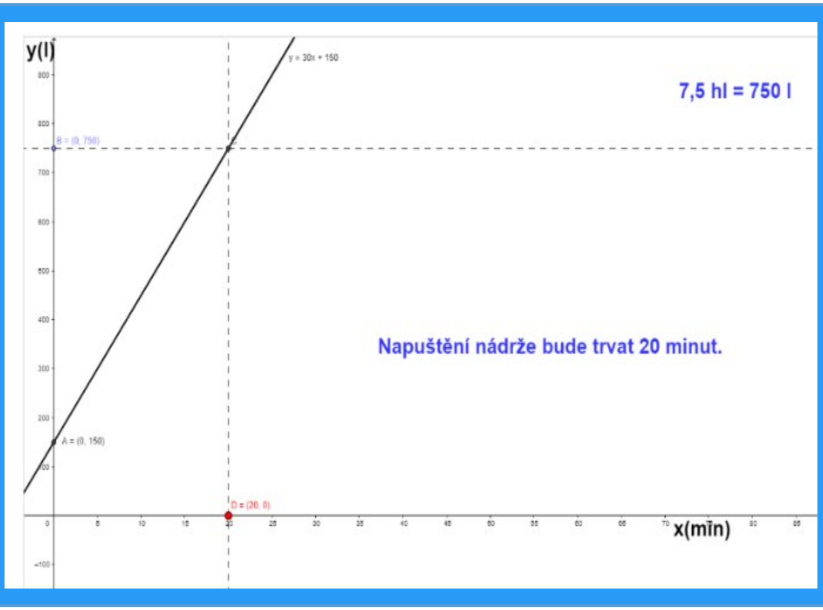


Misconceptions, errors and difficulties in teaching functions

3. Domain and range

- confusion of the value domain and the definition domain
- misunderstanding how the definitional scope affects the value scope
- ignoring the definitional scope in contextual tasks

Example:



7,5 hl = 750 l

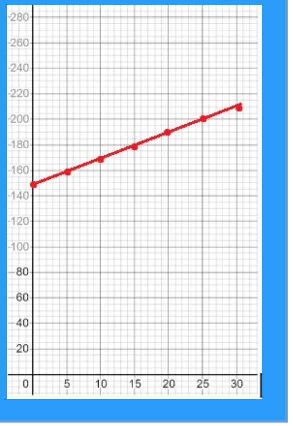
Napuštění nádrže bude trvat 20 minut.

Misconceptions, errors and difficulties in teaching functions

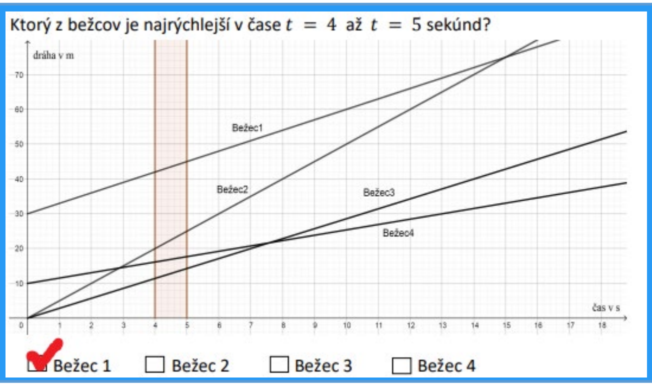
4. Difficulties with graphical representation and coordinate system

- representation or interpretation of continuous data in a discrete way or vice versa
- interval and point substitution
- confusion of slope and height
- a graph as an image
- difficulties in setting two axes for a Cartesian coordinate system
- scaling problems
- the effect of changing the scale of the axes on the appearance of the graph
- confusion of two axes of a graph
- misunderstanding the meaning of points in the same position relative to one of the axes
- points on the graph remain in the same position even if the axes change
- graphs always pass through (or start at) the origin
- the largest numbers marked on the axes represent the largest values reached

Example:
Graphically represent the dependence of the volume in the measuring cup on the number of beads.



Example:


Ktorý z bežcov je najrýchlejší v čase $t = 4$ až $t = 5$ sekúnd?

☑ Bežec 1 ☐ Bežec 2 ☐ Bežec 3 ☐ Bežec 4

Misconceptions, errors and difficulties in teaching functions

5. Difficulties with formula and variables

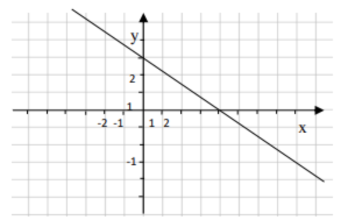
- misunderstanding the difference between coefficients and variables (visible when fitting point coordinates to the general form of a given function)
- algebraic problems when modifying a prescription
- changing the symbol of a variable in a functional equation changes some critical aspects of the function
- misunderstanding the meaning of coefficients (e.g., directives)
- for a linear function:
  - intercepts as coefficients
  - the order of values in the verbal description affects the order of coefficients in the prescription
  - oconfusion between the coefficients *k* and *q*

Example:
The taxi service charges a basic fee of €2.50 and €0.80 per kilometre travelled.

$y = 2.5x + 0.8$

Example:

Daný je graf funkcie *f*:



$y = 4x + 3$

Misconceptions, errors and difficulties in teaching functions

6. Lack of understanding of different aspects

- **Input Output**
  - e.g. problems when filling in a table and reading data from it
- **Covariance**
  - e.g. ignoring covariance properties of individual functions
- **Correspondence**
  - e.g. inability to generalise a relationship using a prescription
- **Object**
  - e.g. difficulty in understanding the properties of functions
  - e.g. failure to identify the type of a function by its prescription if the prescription is not in its basic form