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Lesson Plan Teacher Course

Introduction to Functional Thinking – Aspects of Functions – Functional Thinking in Primary School	
Teaching time:	120 minutes
Target group:	Pre- and In-service teachers
Important links	
Learning environment:	• https://teacher.desmos.com/activitybuilder/custom/58797d35d81a612605304b1f?collections=featured-collections%2C5da6476150c0c36a0caf8ffb • https://www.geogebra.org/m/krebtbds
Description	
Goals:	• Identify the different aspects of functional thinking • Identify and design learning opportunities for its development in primary school
Structure:	• Introduction to Functional Thinking • Create a graphic representation • Use the liquid volume app to sketch graphs • Discussion on how functional thinking is addressed in the Cyprus Mathematics textbooks

This material is provided by the FunThink Team, responsible institution: Team of Mathematics Education – Department of Education University of Cyprus

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Activities

Introduction task 1. (15 min)

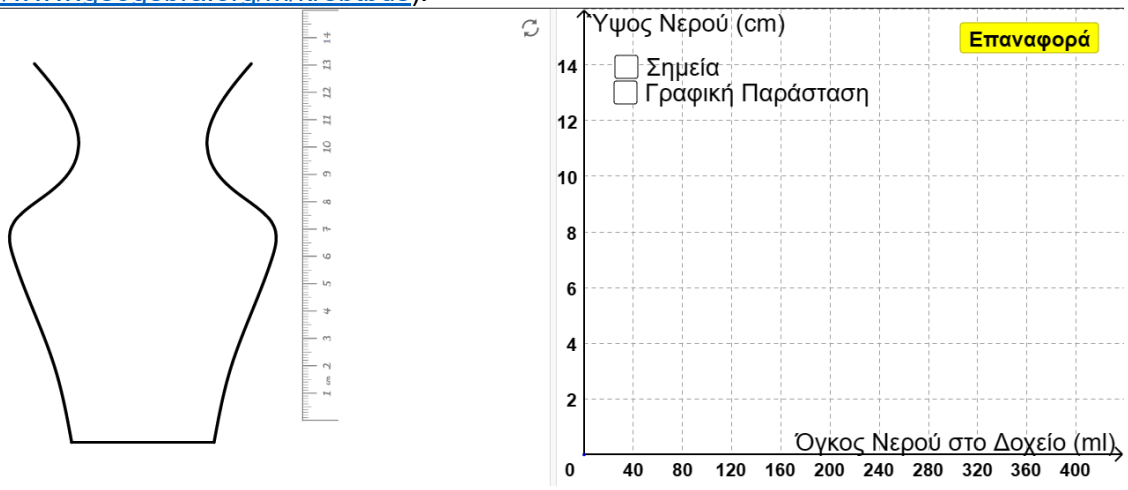
Teachers are shown an animation of a real-life scenario with covarying quantities from the Desmos (Graphing stories) website (https://teacher.desmos.com/activitybuilder/custom/58797d35d81a612605304b1f?collections=featured-collections%2C5da6476150c0c36a0caf8ffb).

Teachers are asked to:

- identify the quantities that vary in the given situation
- suggest possible relations between the identified quantities
- construct a graphic representation of the distance of the movement of the person as shown in the video
- discuss the shape of the graph

Introduction task 2. (15 min)

Teachers are asked to work in the digital form of the Vessels activity (https://www.geogebra.org/m/krebtbds).



Teachers are asked to:

- draw the shape of a vessel based on a given graph
- pose questionw based on the scenario appropriate for students
- compare the questions in terms of processes, content, and cognitive demand
- sketch the shape of a vessel based on the graph

Presentation (15 min)

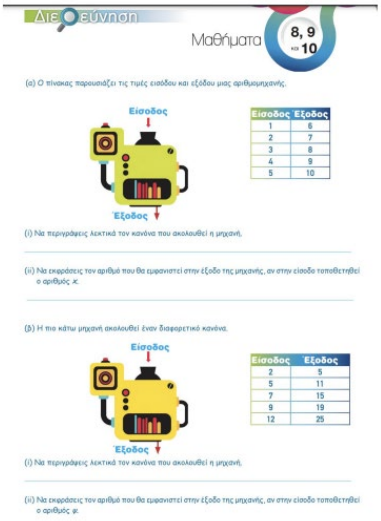
The instructor gives a presentation (Presentation 1) to introduce the teachers to the concept of functional thinking and specific areas such as:

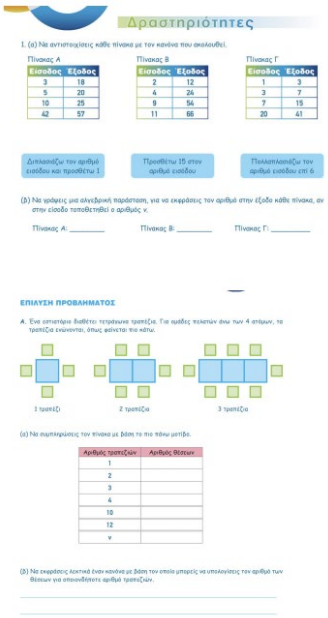
1. Components of functional thinking
2. Definition of functional thinking
3. Aspects of function

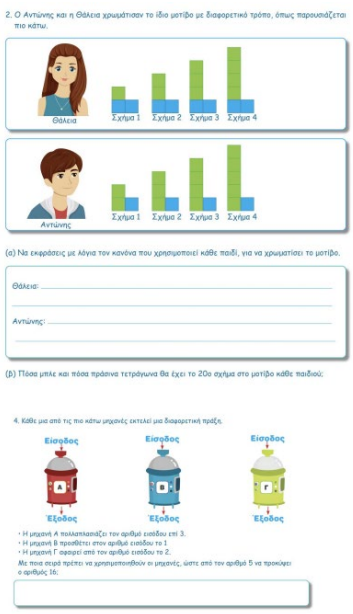
4. Description of functional thinking
    a. For primary school students
    b. For middle school and high school students
5. Functional thinking and algebraic thinking

Whole class - Discussion (25 min)

The participants study activities from the school textbooks (shown below) and discuss to determine which of the aspects of the concept of function are involved in these activities.







Brainstorming in whole class (20 min)

Teachers are asked to pose questions based on the following scenario that address the different aspects of function.

Scenario - Renting a bike: A bike renting company charges 7 euro for a non-refundable deposit and 3 euro for each additional hour.

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Lesson Plan Teacher Course – Meeting 2

Concept of Function as Input-Output and Correspondence	
Teaching hours:	120 minutes
Target group:	Pre- and In-service teachers
Important links	
Learning Environment:	• https://gizmos.explorelearning.com/index.cfm?method=cResource.dspView&ResourceID=1035 • https://www.geogebra.org/m/e4zuj5ss#material/sxq5yzyk • https://www.geogebra.org/m/vsgqkkz3 • https://www.geogebra.org/m/m7bn4s9j • https://app.dwo.nl/embod/?locale=en&profile=108&hash=%23s%3A703843#s:703843
Description	
Goals:	• Identify quantities that vary • Formulate learning opportunities to engage students with the concept of function as an input-output and correspondence • Express that correspondence rule between variable quantities • Interpret representations of functions and make connections between representations (movement, number line, tables)
Structure:	• Introduction task “Walking the number line” • Introduction task “Guess my birthday” • App exploration “Function Machines” • Exploration tasks through GeoGebra app • Activities with the “nomogram” app • Creation of a realistic scenario for given graphic representation

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Activities

Introduction Task 1. (20 min)

Teachers work with a number line placed on the floor from 0 to 100. Specifically, they are divided into groups of two, where one of the teachers places the numbers on the number line and the other finds the rule of the pattern. The instructor then asks the following questions to prompt a whole group discussion:

a. What elements of functional thinking are involved in the above activity?
b. What are the design principles based on which the activity was designed?

Introduction Task 2. (15 min)

Teachers are asked to follow a series of instructions in order to calculate the number resulting from their date of birth.

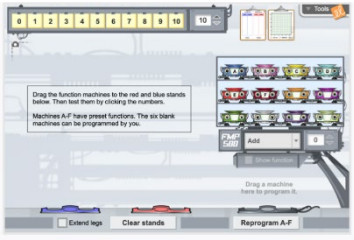
- Write down the number corresponding to your month of birth
- Multiply by 5
- Add 7
- Multiply by 4
- Add 13
- Multiply by 5
- Add the number corresponding to your birthday
- What is your result?

Then, they report these numbers to the instructor, who finds the date of their birthday (5 is subtracted from the last two digits of the calculated number to find the day and 2 is subtracted from the other two digits to find the month). After that, teachers discuss which aspects of functional thinking are involved in the particular activity.

Discussion – App exploration “Function Machines” (30 min)

Teachers explore "Function Machines" app from the Gizmos platform, using a tablet, trying to find out how it works. Afterwards, the instructor initiates a whole-group discussion guided by the following questions:

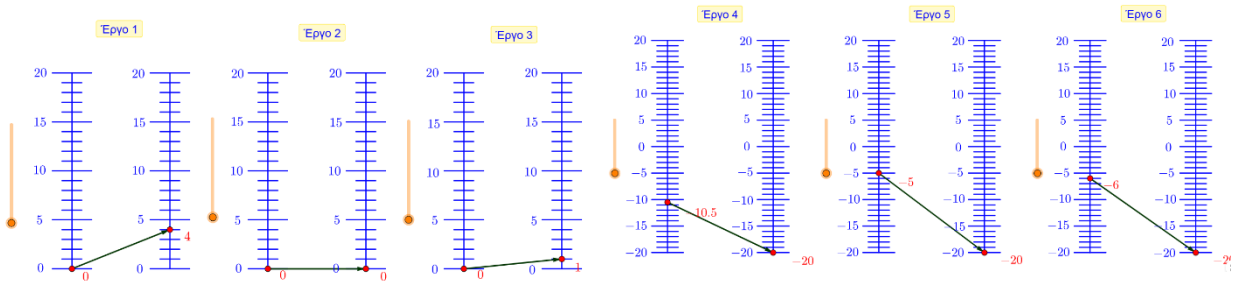
1. In what ways the app could be used to develop students’ functional thinking?
2. What kinds of relationships are involved?
3. What aspect of function could be developed when combing machines?
4. What types of representation could be used?
5. Suggest 3 activities, from easy to difficult, for students to identify different types of relationships between quantities.
6. Suggest realistic scenarios that could be modelled using a single machine or combination of machines.



Discussion – App Exploration ‘

The teachers work on two applicatio s designed in GeoGebra and they discuss which aspects of the co veloped through this learning

environment (Functions Machines (Tasks 1, 2, 3) – GeoGebra , Functions Machines (Tasks 4, 5, 6) – GeoGebra). Teachers are also asked to formulate questions, which they could ask their students, in order to study the direction and magnitude of the covariation in each case.

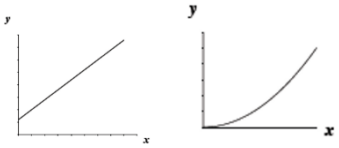


Whole class - Discussion (15 min)

Teachers are invited to watch a video in which the application “Nomograms” (Embodiment (dwo.nl)) is explained. After experimenting and understanding how it works, they discuss in whole-class what the main design principle is behind it, as well as how they could make use of the tool themselves.

Closing Activity (20 min)

As a final activity, teachers are given a worksheet on which there are various graphs (two of these graphs are shown on the side). Teachers are asked to write realistic scenarios based on the graphic representations and discuss difficulties that students may face.



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Lesson Plan Teacher Course – Meeting 3

Concept of Function as Input-Output and Correspondence	
Teaching hours:	120 minutes
Target group:	Pre- and In-service teachers
Important links	
Learning Environments:	• https://tim-lutz.de/funktionenlaufen/indexLinear.html • https://student.desmos.com/join/t29wfg
Description	
Goals:	• Investigate the variation of quantities, identify correspondence relationships and generalize them (verbally and symbolically) • Interpret representations and make connections between these representations (motion, graph)
Structure:	1. Warm Up Activity a. "Walking the Graph" based on given instructions or graphs b. Using a motion capture app 2. Exploration of the "Turtle Crossing" app. ○ Teachers engage in the activities as students 3. Reflection of a given lesson plan and discussion on the aspects of functional thinking students are expected to engage with. 4. Construction of graphs based on realistic scenarios. 5. Watching students interview and discussing the difficulties they faced. 6. Ordering activities based on the level of difficulty. 7. Suggesting scenarios that can be used in primary school to develop students' functional thinking (specifically the covariation aspect of functional thinking)

This material is provided by the FunThink Team, responsible institution: Team of Mathematics Education – Department of Education University of Cyprus

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Activities

Introduction – Warm Up Activity (20 min)

The instructor explains to the teachers that they would work on a number line from 1 to 100 and asked them to get into pairs. Each pair has a card with instructions or graph, which they must represent by moving on the number line. During this process, the other teachers have to watch the pair in order to find out what kind of instructions do they follow and draw the respective graph to represent the route. Next, the instructor presents to the teachers a motion capture application, which records a person's motion and converts it into a distance-time graph.

Exploration. (15 min)

The teachers are asked to work on the "Turtle Crossing" application, from the Desmos website. First, they explore how the application work and then, they follow a series of activities, similar to the ones given to students in primary school.

Whole class - Discussion. (15 min)

The instructor asks the teachers to read the unit “Distance-Time” which was designed by the FunThink Team for primary school. Then, teachers are asked to discuss the following questions:

1. What are the basic design principles of the unit?
2. What aspects of function are involved?

Activity 1. (10 min)

Teachers are asked to construct the graphs for the following scenarios related to the turtle app's context.

Scenario 1: The turtle moves away from the water. Suddenly it takes a short break. Then, it continues to move away from the water.

Scenario 2: The turtle moves away from the water. Suddenly it takes a short break. Then, it starts returning to the water. Before reaching the water, it decides to move away again.

Scenario 3: The turtle moves 8 ft away from the water in 4 seconds. It stops for 2 seconds. Then it returns to the water in just 2 seconds.

Scenario 4: The turtle is 4 ft from the water. It moves another 2 ft away from the water over the next 4 seconds. Then, it returns to the water, traveling 2 ft every second.

Whole class - Discussion (20 min)

Teachers watch selected students’ interviews. In the interviews, students were asked to work on activities with the “Turtle Crossing” app. Based on the video, teachers share ideas on what kind of difficulties students seem to have faced based on their answers and the way they worked.

Activity 2. (25 min)

Teachers are asked to study the following activities and order them according to their level of difficulty. After that, they share their ideas in whole class by referring to the difficulties students may face.

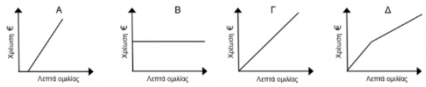
Ο πιο κάτω πίνακας παρουσιάζει την πτήση ενός αερόστατου.

Χρόνος (σε min)	Ύψος (σε m)
0	0
10	120
15	180
30	360
40	410
45	485
60	710
70	710
80	560
60	260

(α) Ο ρυθμός αύξησης του ύψους του αερόστατου ήταν ο ίδιος σε όλη τη διάρκεια της ανόδου; Να εξηγήσετε.

(β) Σε ποια χρονικά διαστήματα ο ρυθμός αύξησης ή μείωσης του ύψους του αερόστατου ήταν ο ίδιος; Να εξηγήσετε.

Οι γραφικές παραστάσεις A, B, Γ και Δ παρουσιάζουν πακέτα χρέωσης μιας εταιρείας κινητής τηλεφωνίας με βάση τα λεπτά ομιλίας.



Πιο κάτω δίνονται λεκτικές περιγραφές:

Π_1: Δεν υπάρχει μηνιαίο πάγιο. Υπάρχει σταθερή χρέωση ανά λεπτό ομιλίας.

Π_2: Υπάρχει μηνιαίο πάγιο και σταθερή χρέωση ανά λεπτό ομιλίας.

Π_3: Υπάρχει μηνιαίο πάγιο. Προσφέρονται δωρεάν λεπτά χρόνου ομιλίας. Όταν ξεπεραστούν τα λεπτά αυτά, υπάρχει σταθερή χρέωση ανά λεπτό ομιλίας.

Π_4: Δεν υπάρχει μηνιαίο πάγιο. Υπάρχει σταθερή χρέωση για έναν αριθμό λεπτών ομιλίας. Όταν ξεπεραστούν τα λεπτά αυτά η χρέωση ανά λεπτό ομιλίας μειώνεται.

Π_5: Υπάρχει μηνιαίο πάγιο και προσφέρεται δωρεάν απεριόριστος χρόνος ομιλίας.

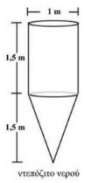
Π_6: Δεν υπάρχει μηνιαίο πάγιο. Προσφέρονται δωρεάν λεπτά χρόνου ομιλίας. Όταν ξεπεραστούν τα λεπτά αυτά, υπάρχει σταθερή χρέωση ανά λεπτό ομιλίας.

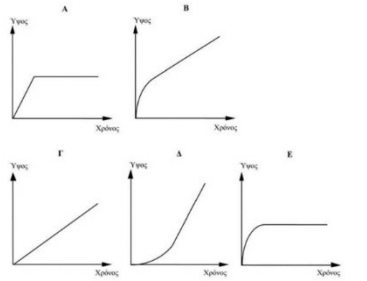
Να αντιστοιχίσετε τις γραφικές παραστάσεις με τις λεκτικές περιγραφές, συμπληρώνοντας τον πιο κάτω πίνακα.

Γραφική Παράσταση				
Λεκτική Περιγραφή				

Πίνακας Τιμών	Εξίσωση
x | y -2 | -5 -1 | -3 1 | 1 2 | 3	$y = x$
x | y -2 | 3 -1 | 0 1 | 0 2 | 3	$y = 3$
	$y = 2x - 1$
	$y = x^2 - 1$

11. Ένα ντεπόζιτο νερού έχει τη μορφή και τις διαστάσεις που φαίνονται στο διπλανό σχήμα. Αρχικά το ντεπόζιτο είναι άδειο. Μετά το γεμίζουμε νερό με ρυθμό ένα λίτρο ανά δευτερόλεπτο. Ποια από τις παρακάτω γραφικές παραστάσεις δείχνει πώς το ύψος του νερού μεταβάλλεται με την πάροδο τού χρόνου;





PISA 2003

Closing (15 min)

The instructor encourages teachers to suggest scenarios that can be used in primary school to help students develop understanding about the function as covariation.

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Lesson Plan Teacher Course – Meeting 4

Patterns	
Teaching hours:	120 minutes
Target group:	Pre- and In-service teachers
Important links	
Learning environment:	• https://www.geogebra.org/m/st85nctn • Video (2.50'): https://www.youtube.com/watch?v=t179ZcUdCOA&t=241s • https://www.geogebra.org/m/u8jaspkc • https://www.geogebra.org/m/se9dk6sc • https://www.geogebra.org/m/q5ubbc9g • https://www.geogebra.org/m/tjgrwmkp • https://teacher.desmos.com/activitybuilder/custom/61e8507ff070d2108c7bb984?collections=61bcc95700581818dff1d4d7%2C61bcca1bdb50d15651f028c4#preview/f83a40d7-bad1-44aa-a191-e64e564cf6a3 • https://apps.mathlearningcenter.org/pattern-shapes/
Description	
Goals:	• Experience the aspects of function as found in pattern activities: input-output, correspondence and covariation. • Interpret representations, make connections between representations (graphical representation of patterns, tables, graphs). • Generalize the relationship between quantities (verbally and symbolically).
Structure:	1. Introduction to patterns using the app https://www.geogebra.org/m/st85nctn 2. "Human pyramid" • Discussion in whole class https://www.youtube.com/watch?v=t179ZcUdCOA&t=241s 3. Exploration of the app

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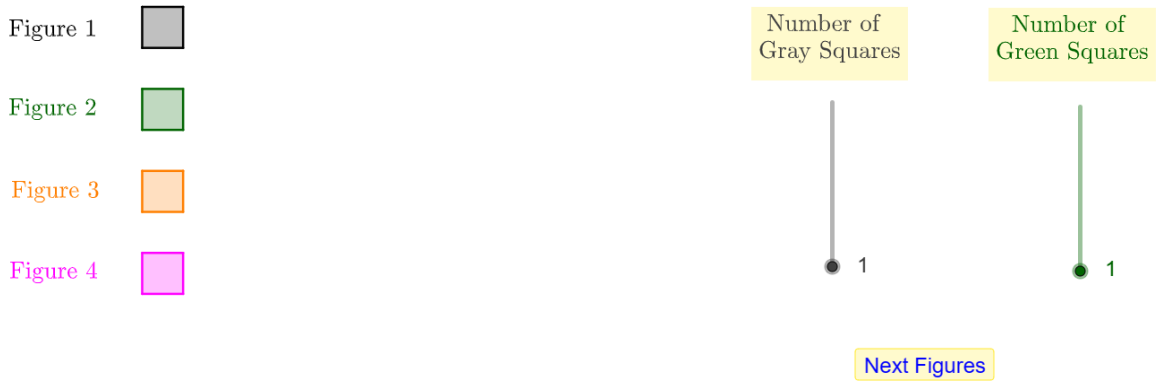
	• Pose questions to engage students with the different aspects of functional thinking https://www.geogebra.org/m/u8jaspkc 4. PowerPoint Presentation on patterns • Definition • Pattern categories and types of patterns • Suggested activities 5. Exploration of apps • https://www.geogebra.org/m/se9dk6sc • https://www.geogebra.org/m/q5ubbc9g • https://www.geogebra.org/m/tjgrwmkp 6. Work on handout and whole-class discussion 7. Video of classroom episode • Reflection on the classroom episode 8. Final activity

Activities

Introduction task 1. (10 min)

Teachers are asked to explore the pattern app in GeoGebra (https://www.geogebra.org/m/st85nctn) and to identify what happens when they change the values of the gray and green squares (using the two sliders). Then, they share ideas to find the recursive rule of the pattern (i.e., the number of the squares in each next figure equals the sum of the number of squares in the previous two figures minus one).



Whole class discussion. (15 min)

The instructor shows a video. The video shows a group of students who create a human pyramid. Teachers are encouraged to discuss how the human pyramid was created. Then, the instructor asks them to share their thoughts on how the video could be used in their teaching, what kind of questions would they pose to students, what the expected answers would be, and in what ways the human pyramid video could help students in understanding what a pattern is.
Questions:

1. How could the video be used in the teaching of patterns?
2. What questions could you use to engage students in identifying relationships? (input-output, covariations, correspondence)

Activity 1. (10 min)

Following the previous whole-class discussion, the instruction shows another app in GeoGebra (https://www.geogebra.org/m/u8jaspkc). Teachers could create pyramids of different sizes using the app (similarly to the video with the human pyramid). Teachers are asked to explore the app and pose questions that could be used in classroom to encourage students in thinking about input-output quantities, correspondence and covariation relationships.

Presentation. (15 min)

The instruction uses a PowerPoint presentation and explains (a) what a pattern is by presenting the formal definition, (b) different criteria of categorizing patterns (e.g., based on alternation of shape, size), (c) different types of patterns based on the structure of the pattern (e.g., AB, AAB), and (d) activities of varying difficulty to develop students' competencies related to patterns.

Activity 2. (10 min)

Teachers are asked to explore three pattern apps in GeoGebra and to describe the patterns in different ways (e.g., how the pattern grows, how to find each next term, how the figure number could be used to find the number of squares) (https://www.geogebra.org/m/se9dk6sc, https://www.geogebra.org/m/q5ubbc9g, https://www.geogebra.org/m/tjgrwmkp).

Activity 3. (30 min)

The instructor asks the teachers to work on a handout with different tasks. Teachers are given time to work on the tasks and then they are asked to discuss what kind of difficulties students may face and what kind of questions would they use.

Task 1:

In what ways the concept of function could be seen in patterns? Find a function rule that corresponds to the following pattern.

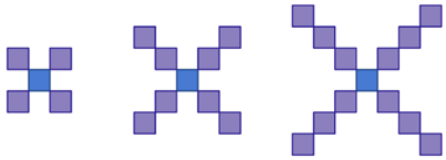
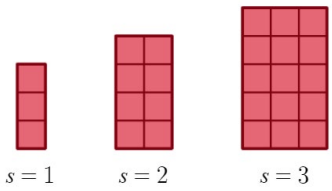


Figure 1 Figure 2 Figure 3

Task 2:

The following students describe the pattern below.



- Paul: “I see 1 square and two rows”
- Liz: “I see 1 rectangle for which one side is 2 units more than the other side”

(a) Explain how each student ‘sees’ the pattern.

(b) How would each student describe the figure for s=4;

Task 3:

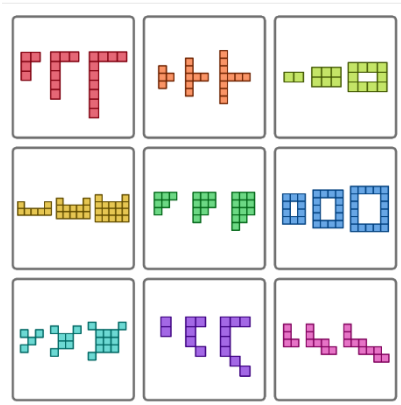
Select one of the following patterns.

(a) Draw the next figure.

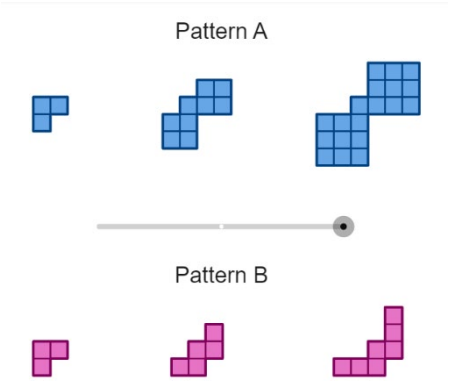
(b) Draw Figure 10.

(c) Explain how to find the number of squares for Figure 10.

(d) Find a mathematical sentence that could be used to find the number of squares for Figure n.



Task 4:



(a) In what ways the two patterns above are similar? In what ways do they differ?

(b) How many squares would each pattern have in Figure n?

Whole class discussion. (20 min)

The instructor shows a video of a classroom episode from a sixth-grade classroom. After watching the video, the teachers reflect on how they would approach the same activity in class. Furthermore, the teachers share ideas and give feedback on how the apps they worked on during the session could be used in their classroom.

Final activity. (10 min)

The teachers work on the following final activity:

(a) Draw two different options for Figure 3.

Figure 1 Figure 2

(b) Draw a pattern that corresponds to the following table.

Figure Number	Number of pieces
1	4
2	7
3	10