Co-funded by the
Erasmus+ Programme
of the European Union

FUNTHINK MANUAL TEACHER EDUCATION

MODULE 1: FUNCTIONAL THINKING

Overall learning objectives of pre-service teacher education:

1. **Quantitative reasoning:**
    - What are the quantities that vary
    - How do the quantities co-vary
    - How do they correspond
    - Object view
2. **Representations of functions (adaptive expertise/fluency)**
    - Moving between representations (table/graph/numbers/map/story)
    - Linking representation and situation

These learning objectives come to the fore in three modules:

1. Variation-co-variation
2. Distance-time graphs
3. Patterns

Each of these modules consist of a learning trajectory with the following characteristics:

- Students experience covarying quantities
- Identify co-varying quantities; track them
- Explain how they relate
- Describe the correspondence

Key principles with each of these modules are variables, relationships, and generalization; key processes are noticing, describing, justifying, representing, generalizing, reflecting and applying.

In this document you can find the **teacher manual for pre-service teacher education**. This module consists of three main activities:

1. Walking the number line and describing the experienced relations
2. Walking in front of a motion sensor and produce distance-time graphs
3. Explore the relationship between distance-time graphs and movement in a digital environment

Lesson Plan Module pre- and in-service Teachers

Module:	Functional thinking
Teaching Hours:	2x 60 min (for pre-service teachers): Lesson 1 - General introduction in FunThink (10 minutes) - Walking the number line (25 minutes) - Reflection (5 minutes) - Double numberline app (10 minutes) - Reflection + whole class discussion and conclusion (10 minutes) Lesson 2 - Review last lesson (5 minutes) - Walking in front of a sensor (25 min) - Reflection (5 minutes) - Walking Turtle app (10 min) - Reflection + whole class discussion (15 minutes)
Grade Level/Age Range:	(Pre- and in-service) teachers
Brief Description:	In this module, (pre-service) teachers investigate functional thinking through two activities, researching the relationship between input and output numbers whilst 'walking on the number line', and investigating distance-time graphs. The pre-service teachers use real-world experiments to understand functional thinking. The pre-service teachers also experiment with the digital tools, connected to the earlier described activities. The focus of this module is on developing and promoting a qualitative understanding of functional relationships and how to teach them. The module engages pre-service students with the relationship between distance and time, explore the graphical representation as well as the rate of change in distance/time scenarios.
Design Principles:	**Inquiry** ■ ■ **Situatedness** ■ ■ **Digital tools** ■ ■ ■ **Embodiment** ■ ■ ■
	- Inquiry based learning: Students explore and find out which movements created/creates which kinds of graphs; Students find out how the content of a graph relate to movement of the turtle; - Situatedness: Students see a direct representation of their own movement (or the movement of the turtle); - Digital: transfer from physical activities to a digital activity, from a motion sensor to an application - Embodiment: Students connect their own physical movement (or the turtle's movement) to the formal representation of the distance-time graphs.

Functional Thinking:	**Input – Output**			
	Covariation			
	Correspondence			
	Object			
Learning Goals:	- Pre-service teachers experience covarying quantities and explain how they relate - Pre-service teachers learn to move between different representations of a function - Pre-service teachers reflect on the concept of function and the importance of developing functional reasoning - Pre-service teachers reflect on the relationship between the task and the existing curriculum and the design principles underlying the task			

Lesson no. 1. Physical activities

Activities

Activity 1.

Introduction to functional thinking

This introductory activity requires the whole class to participate.

Introduction

The teacher introduces the lesson by providing the pre-service teachers some information about the FunThink project. The teacher shows some examples of functional thinking and mentions the design principles used in the developed activities. The pre-service teachers are asked what their prior knowledge is about the subject and whether they have taught functions before.

Note: if the pre-service teachers need to do the pre-test assessment at the start of this lesson, the introduction should follow the pre-test, to not influence the findings.

Estimated duration: 10 minutes

Activity 2. Walking the number line & Walking graphs

2a. Walking the number line

This activity will be performed by half of the group of students. The other half will perform activity 2b.

Introduction

The teacher places the number line in front of a group of pre-service teachers, preferably not in the classroom, and explains the activity. The teacher demonstrates an example with volunteers. The teacher explains that pre-service teachers will work in pairs. One pair will get a card with a secret rule (an operation, e.g. "+2") on it. The other pair will need to work out what the secret rule is. by walking along and standing on the number line.
Then, as an example, the group of pre-service teachers plays the game once. The teacher shows a card with a rule (e.g., '+2'; always start with a simple addition problem) to all pre-service teachers but not to himself/herself. The teacher must discover the rule. The teacher stands on the number '1' on the number line and sticks a sticky note on that number. The pre-service teachers (who know the secret rule) walk to the outcome of their secret rule (When '1' is the input for the calculation, they stand on '3'), and place a sticky note on that number. The teacher then reasons out loud to figure out what the secret rule could be. The teacher proposes different possible rules but needs more information to be sure. This is a step like data gathering and forming hypotheses. The teacher then walks to another number on the number line, which leads to the pre-service teachers walking to the new output number (on both numbers a new sticky note is stuck). This is a step of verification. The teacher then explains how they know the rule and shares it with the pre-service teachers.

The teacher explains that the roles of the pairs will be switched around each time: after discovering a secret rule, the roles switch and the other pair of pre-service teachers has to discover the rule. The pre-service teachers are divided into groups of four (two pairs) and play the game together. After each pair of pre-service teachers has played both roles (knowing and discovering), the group of four together make a poster of their findings. The poster reflects the reasoning of the pre-service teachers.

Suggested tools/materials:

- Two (or three) physical number lines from 1-100, so that about 12-15 pre-service teachers can work on them simultaneously. Preferably, use a number line at which all number up to 100 are visible to clearly visualise process of discovering the 'secret rule'.
- Cards with the 'secret rules' on them. Addition (+1 until +9), subtraction (-1 until -9), multiplication (x2 until x5) and division (:2 until :5)
- Sticky notes in different colours
- Blank A3 sheets
- Pencils in different colours

Estimated duration: 5 minutes

Walking the number line

After the whole class introduction, the pre-service teachers perform the activity. In small groups, they take turns to knowing or discovering the rule. After each pair of pre-service teachers has played both roles (knowing and discovering), the group of four together make a poster of their findings. The poster has to reflect the reasoning of the pre-service teacher.
The teacher walks around to guide the pre-service teacher. The teacher hands out the cards in the following order:

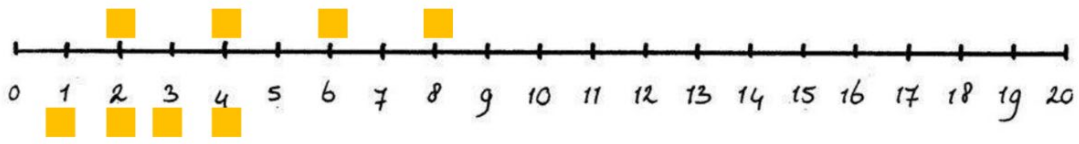
1. Singular addition or subtraction problems (e.g., +2, -4).
2. Simple singular multiplication problem (e.g., x2, x3)
3. Simple composed problems, combining addition and multiplication (e.g., x2 +1)
4. More difficult composed problems, combining all types of operations (e.g., :3 – 4)

Moreover, the teacher walks around, observes, and asks questions like:

- Are you sure? Why (not)?
- How can you find out?
- How do you know?
- Why do you choose this number?

The teacher also encourages pre-service teachers to make a poster of their findings.

By using the sticky notes, the pattern of the input-output combinations becomes visible for pre-service teachers and teacher to inspect:



Suggested tools/materials:

- Two (or three) physical number lines from 1-100, so that about 12-15 pre-service teachers can work on them simultaneously. Preferably, use a number line at which all number up to 100 are visible to clearly visualise process of discovering the ‘secret rule’.
- Cards with the ‘secret rules’ on them. Addition (+1 until +9), subtraction (-1 until -9), multiplication (x2 until x5) and division (:2 until :5)
- Sticky notes in different colours
- Blank A3 sheets

- Pencils in different colours

Estimated duration: 15 minutes

Activity 2b: Walking a graph

This activity will be performed by half of the group of students. The other half will perform activity 2a.

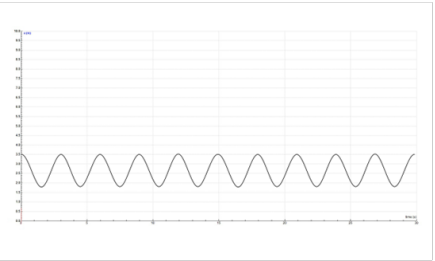
Introduction

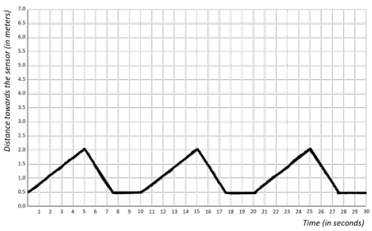
The teacher introduces the activity by asking one of the pre-service teachers to come to the front, to stand and/ or perform movements (walking forward/backward, standing still in one position …) on the line on the floor. The movement of the pre-service teacher is recorded with the help of a sensor and displayed on the screen. The teacher guides and asks questions:

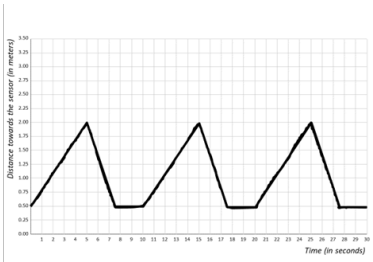
- *How does the graph and the movement correspond?*
- *Can you walk a certain pattern (e.g. zigzag)?*

Walking a graph

The teacher divides the pre-service teachers into small groups. Each group gets two or three example graphs. Pre-service teachers are asked to start with one graph and to devise a 'walking plan' for walking the depicted graphs.







Teacher asks pre-service teachers to create a certain graph by walking. The teacher asks questions to the pre-service teachers:

- *Do the graphs look similar?*
- *What are the differences and why?*
- *How can you adjust your walking in such a way that the graph becomes more like the example? And why?*

The teacher shows the following two graphs and asks the pre-service teachers to describe the graph and the corresponding movement.

Suggested tools/materials/:

- Laptop with Coach 7 software (lite version freely available via: https://cma-science.nl/coach-7-lite_en)

- Motion sensor
- Example graphs

Estimated duration: 20 minutes

Whole classroom discussion

Activity 3. Next lesson

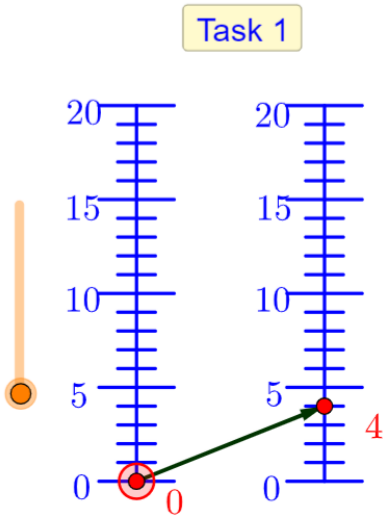
During this lesson, the pre-service teachers will explore and get experience with the developed digital tools. In the first part of this lesson, the pre-service teachers will experience the different digital tools, followed by a whole class reflection on the combination of physical and digital experiences in relation to functional thinking.

Exploration of digital activities

Activity 3A – Double number line

Introduction

The teacher starts by recalling the activity of walking on the number line. What was the goal of the activity (discovering the secret rule)? And in what way could this goal be achieved (trying different values and exploring the effect)?
The teacher then explains that a similar activity will be performed today. Only this time pre-service teachers will not walk themselves. Instead, they can 'create simulate movement' in a virtual environment.
The teacher starts with opening opens the Geogebra application with Tasks 1-3 (https://www.geogebra.org/m/vsgqkkz3). The teacher shows that by dragging the red point on the left axis of the double number line or by dragging the orange button, the numbers on the left axis of the double number line change. Meanwhile, the numbers on the right axis of the double number line also vary. The teacher discusses with the pre-service teachers the differences and similarities between Task 1 in the digital environment and the activity of walking the number line.



Similarities:

- Number line
- You change one value and the other value changes as a response
- …

Differences:

- One number line with values on each side vs. two number lines
- Range of numbers 0-100 vs. range 0-20
- Only whole numbers are visible vs. also decimal numbers

- Only positive numbers vs. also negative numbers (tasks 4-6)
- …

Suggested tools/materials:

- Geogebra application double number line tasks 1-3: *https://www.geogebra.org/m/vsgqkkz3*
- Digiboard to project the application

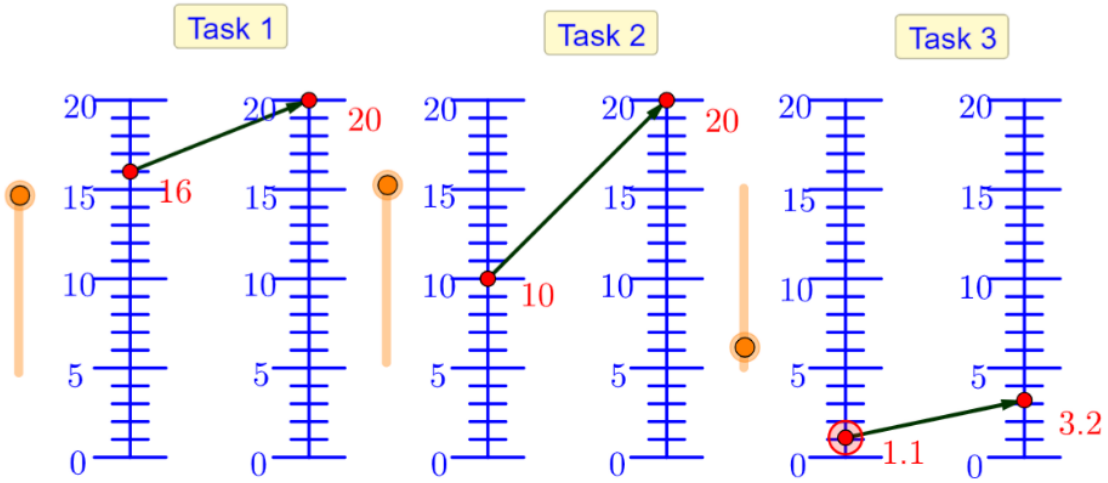
Estimated duration: 10 minutes

Double number line: discover the secret rules

Pre-service teachers then work in pairs with the application. Pre-service teachers are asked to drag the red dot on the left axis of each double number-line and observe how the arrow on the right axis moves. They start with tasks 1-3 and once finished continue with tasks 4-6.

In total, pre-service teachers work on six tasks with the following hidden functions:

Task 1	+4	Task 4	x2 +1
Task 2	x2	Task 5	x3 -5
Task 3	x2 +1	Task 6	x2 -8



Pre-service teachers' task is to discover the secret rules. During the activity, pre-service teachers are asked to keep track of what they discover. They can choose their own representation for this, for example a table or a graph.

The teacher walks around and asks questions with the aim to elicit pre-service teachers' exploration and reflection. Questions such as:

- Are you sure? Why (not)?
- How can you find out?
- What strategies can you use?
- What strategies did you use on the physical number line?
- How do you know?
- Why do you choose this number?

During the activity the teacher constantly refers to the prior experiences with walking the number line.

Suggested tools/materials:

- Tablets for each pair of students
- Double number line tasks 1-3: https://www.geogebra.org/m/vsgqkkz3
- Double number line tasks 4-6: https://www.geogebra.org/m/m7bn4s9j

Estimated duration (in total): 30 minutes

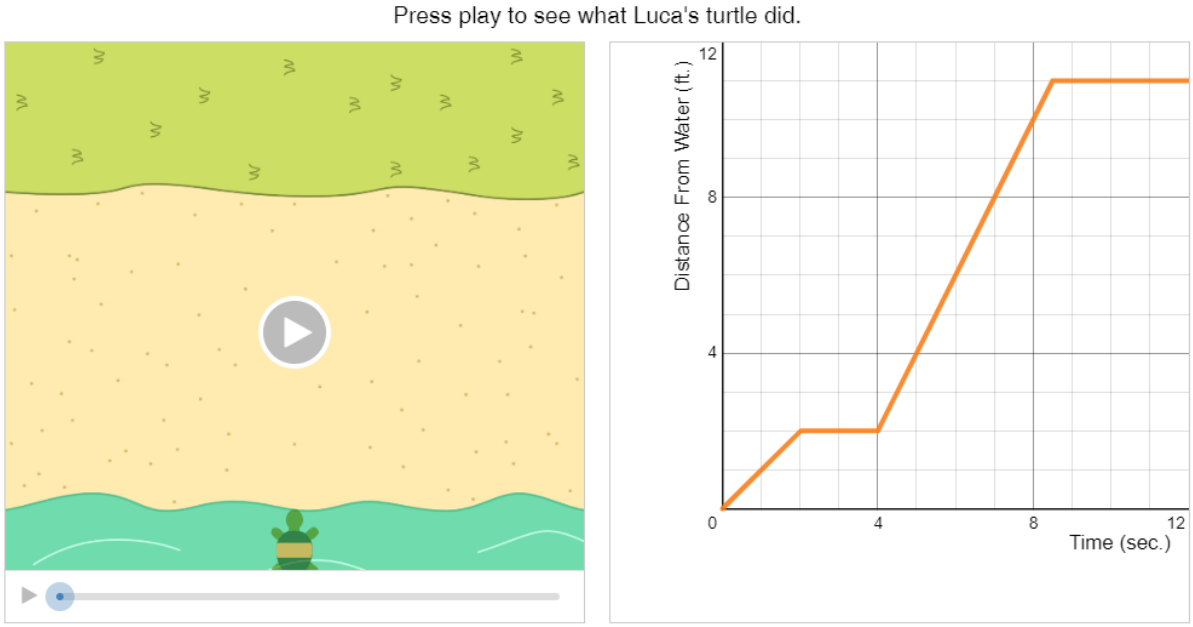
Activity 3B – Distance-time | Turtles

Introduction

The teacher opens the Desmos app in front of the group of pre-service teachers. The teacher draws a graph and shows the corresponding movement of the turtle. The teacher asks the pre-service students to describe the movement. The teacher discusses what kind of quantities are incorporated in the descriptions.

Exploration

The pre-service teachers will now work in pairs on the Desmos app on their devices (preferably tablets), as shown below:



Pre-service teachers are asked to draw points and different sketches and describe how the turtle moves by playing the video.

The teacher could guide the exploration of the pre-service teachers using the following guidelines:

- *Draw points and describe the place of the turtle;*
- *Draw different sketches and describe how the turtle moves;*
- *Draw a line sketch and observe the turtle's journey. Draw a steeper line, how does the turtle's journey change?*
- *Draw line sketches that starts from different points on the y-axis and observe the turtle's journey. How does the turtle's journey change?*

The teacher can also guide the pre-service teachers by asking questions (e.g.):

- *What quantities are shown in the two axes?*
- *How does the turtle journey change?*
- *How does the movement of the turtle relate to a change in the graph?*

- *Can you make an hypothesis of the turtle's journey based on the graph?*
- *Can you cover the graph and make an hypothesis of the graph based on the turtle's journey?*

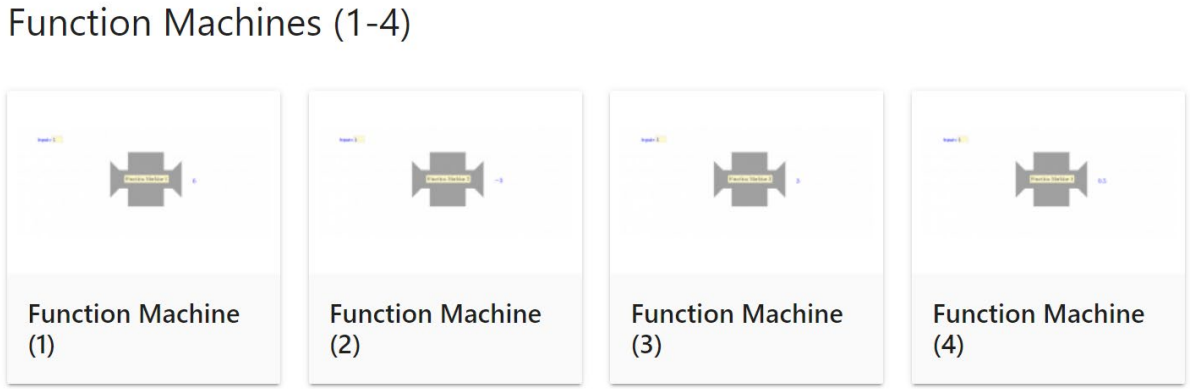
Suggested tools/materials/:

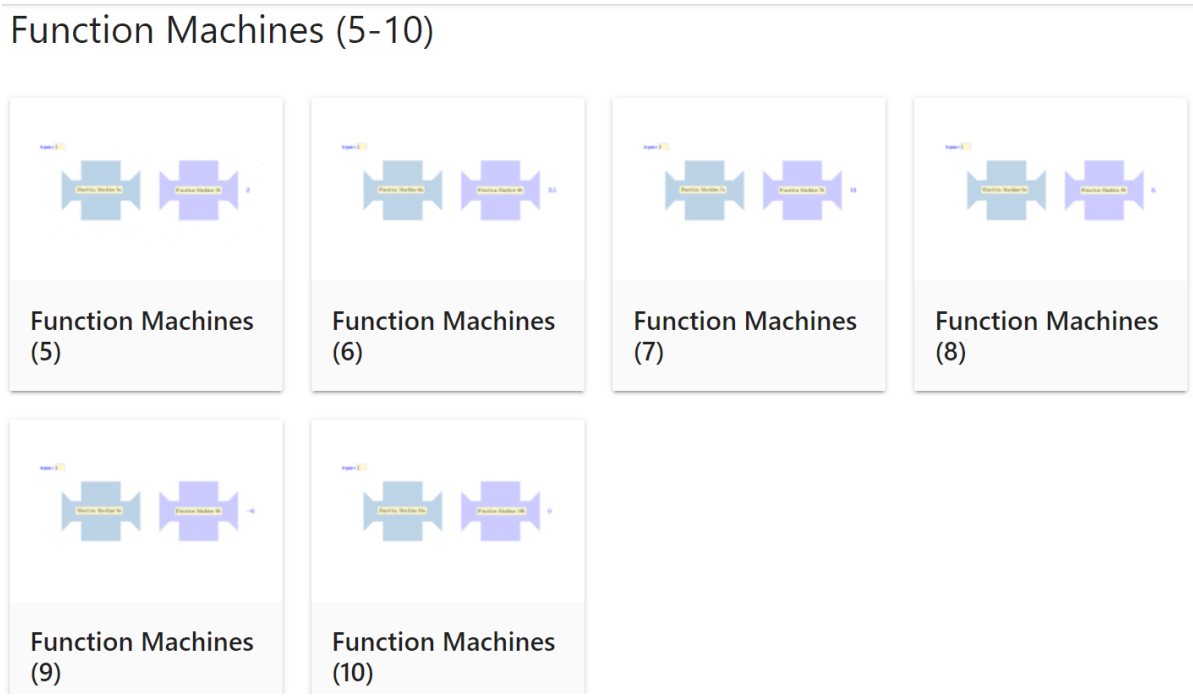
- Tablet devices
- Desmos app
    - EN:https://teacher.desmos.com/activitybuilder/custom/5ddbf9ae009cd90bcdeaadd7?lang=nl&collections=featured-collections%2C5da6476150c0c36a0caf8ffb#preview/8809fa03-a71e-45d9-b2cd-bef8ee337602
    - NL:https://teacher.desmos.com/activitybuilder/custom/5fadcd24785f5f384d942088?lang=nl&collections=featured-collections%2C5fadcd14a8b53c39e12bdc89#preview/8809fa03-a71e-45d9-b2cd-bef8ee337602

Estimated duration (in total): 30 minutes

Activity 3C – Function Machines

Pre-service teachers will look at the different activities about function machines as a digital follow-up from the physical experience of walking the number line. Pre-service teachers preferably work in pairs.





Tasks 1-4 are singular problems and tasks 5-10 are composed problems. In total, pre-service teachers work on ten tasks with the following hidden functions:

Machine 1	+5	Machine 6	:2 +3
Machine 2	-4	Machine 7	+4 x2
Machine 3	x3	Machine 8	x2 +4
Machine 4	:2	Machine 9	-3 x3
Machine 5	+3 :2	Machine 10	x3 -3

During this activity, the teacher walks around, observes, and asks questions like:

- Are you sure? Why (not)?
- How can you find out?
- What strategies can you use?
- What strategies did you use on the physical number line or the double number line?
- How do you know?
- Why do you choose this number?

During the activity the teacher constantly refers to the pre-service teachers' experiences with walking the number line and the experiences in the digital environment with the double number line.

Please note that, since the composed function machines consist of two separate operations, the order in which the function machines execute the operations does not necessarily follow the standard procedures for the execution of mathematical operations. For example, function machine 5 first adds 3 to the input number and then divides the outcome by 2 (in short: +3 :2). According to the standard procedures of the execution of mathematical operations, where division precedes addition, this should be the other way around. Please, discuss this discrepancy with students when necessary.

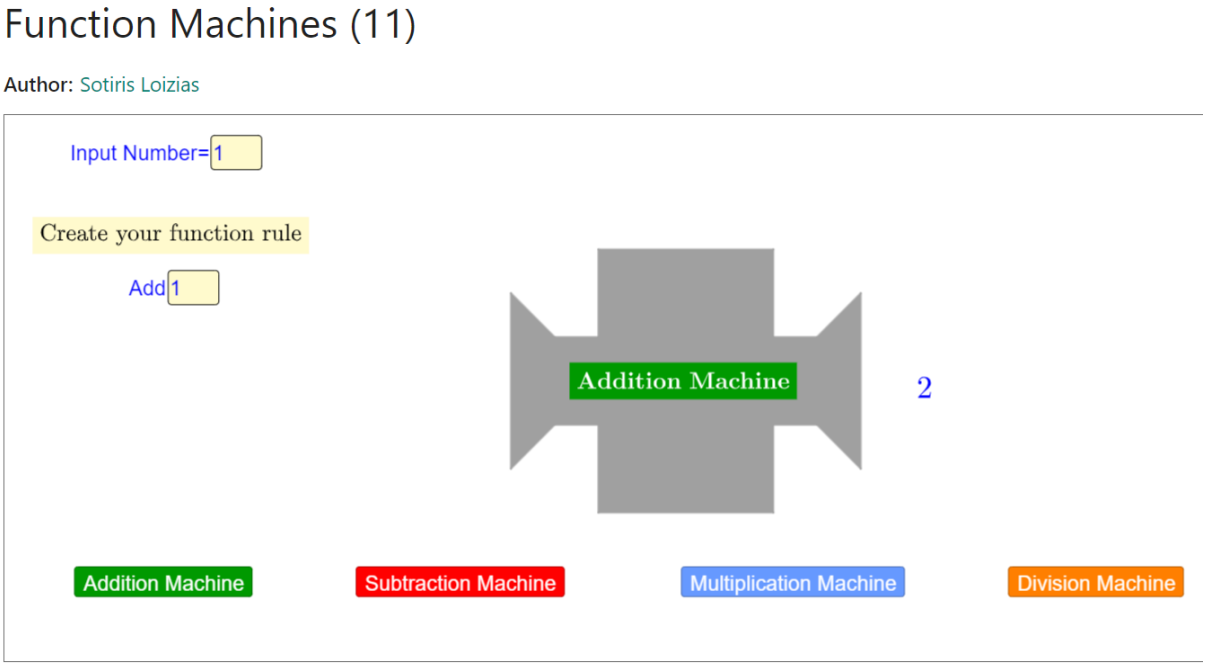
Suggested tools/materials:

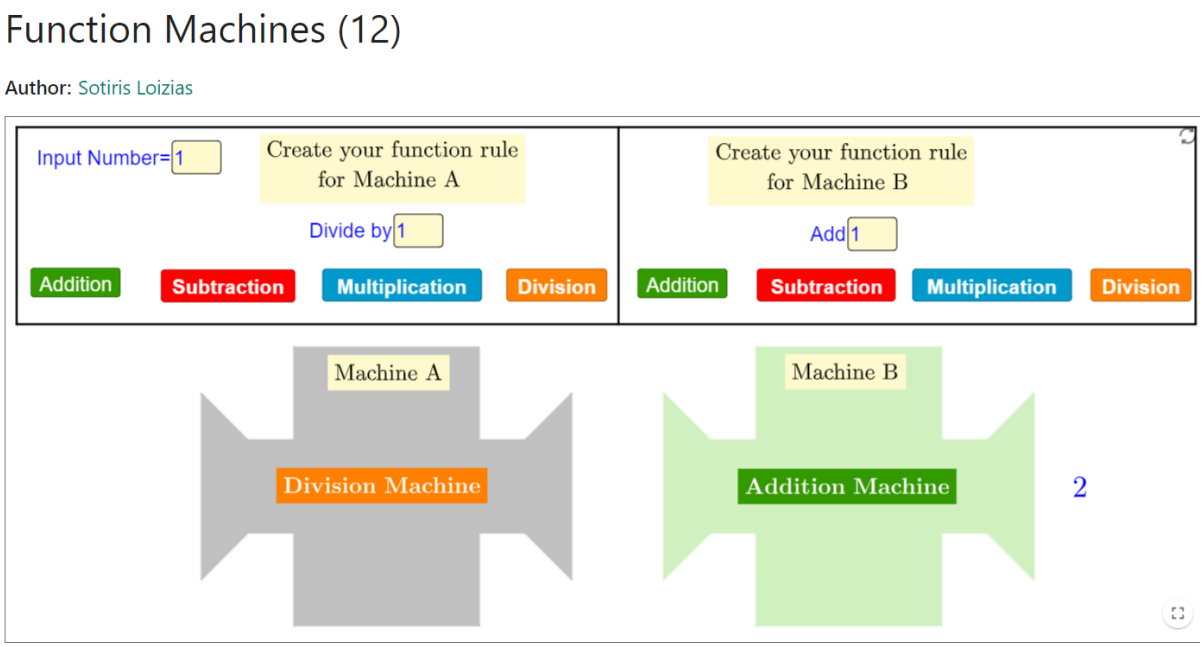
- Tablets for each pair of pre-service teachers
- Function machines tasks 1-10 *(https://www.geogebra.org/m/e4zuj5ss).*

Estimated duration: 15 minutes including classroom discussion

Function machines – Part 2

Introduction: For part 2 of the activity, the teacher shows function machines 11 and 12. The teacher explains that now pre-service teachers can create their own machines. The teacher shows how to change the 'secret rule' of the function machines and explains (in Dutch) what addition (+), subtraction (-), multiplication (x) and division (-) means.





The teacher then writes down the following pair of numbers on the whiteboard:

4	13

Pre-service teachers' task is now to come up with different function machines which have an input value of 4 and an output value of 13 (e.g., +9; x2 +5; x4 -3). Students mention alternatives and the teacher and the pre-service teachers together build the function

machines. After this example, each pair of pre-service teachers gets two pairs of values for which they have to come up with as many function machines as possible.
Task 1:

3	15

Task 2:

4	6

During this activity, the teacher walks around, observes, and asks questions like:

- What strategies do you use?
- How can you find out if this machine is correct?
- Are you sure? Why (not)?
- How do you know?

Suggested tools/materials:

- Tablets for each pair of pre-service teachers
- Function machines tasks 11-12 *(https://www.geogebra.org/m/e4zuj5ss).*

Estimated duration (in total): 30 minutes

Activity 4.

Reflection + assessment activities

After the exploration of both the physical and digital activities, the pre-service teachers will join the whole class reflection on the different activities and digital environments for developing functional thinking. In this whole class reflection, several subjects will be reflected on:

- How the physical activities develop functional thinking
- How the digital activities are related to the physical activities
- How the teacher guided the pre-service teachers during the physical and digital activities
- What guidance of the teacher would be beneficial for students
- How the pre-service teachers will think that their students will respond to these activities
- How the design principles came to the fore in the activities

Based on the answers of the pre-service teachers, the teacher asks deepening questions during this reflection. These above-mentioned questions can guide the reflection.

If necessary, the teacher then concludes the lesson with the assessment.

The teacher can also give the students two assessment activities, to test the current understanding of functional thinking in distance-time scenarios. See the items on the next pages.

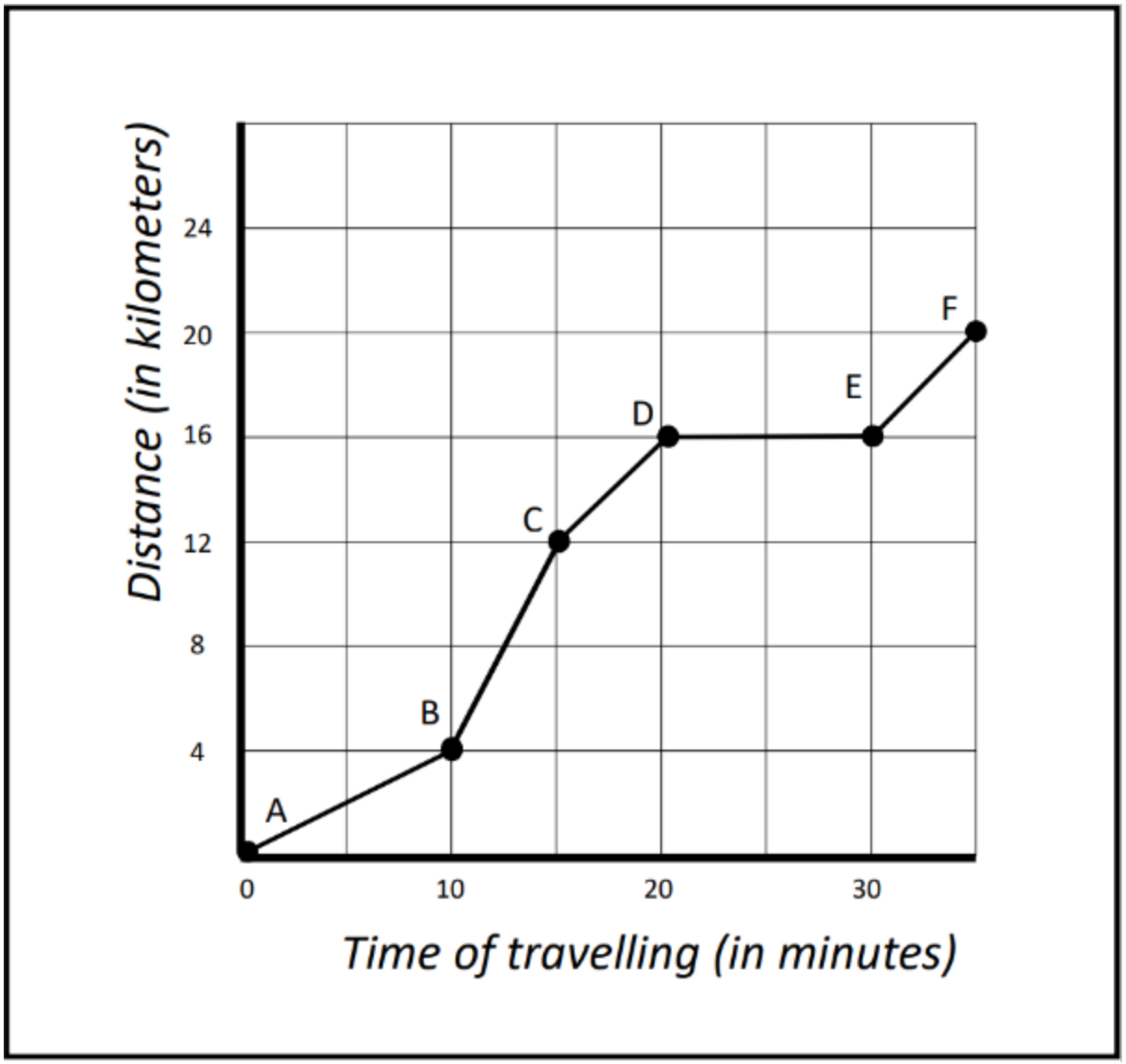
All assessment activities are derived from the study of Duijzer (2020).

Duijzer, C. (2020). *Moving towards understanding: Reasoning about graphs in primary mathematics education* [Doctoral dissertation, Utrecht University]. Utrecht University Repository. https://dspace.library.uu.nl/handle/1874/398915

ASSESSMENT ITEMS

1. **A car drive.**

A car drives through town:

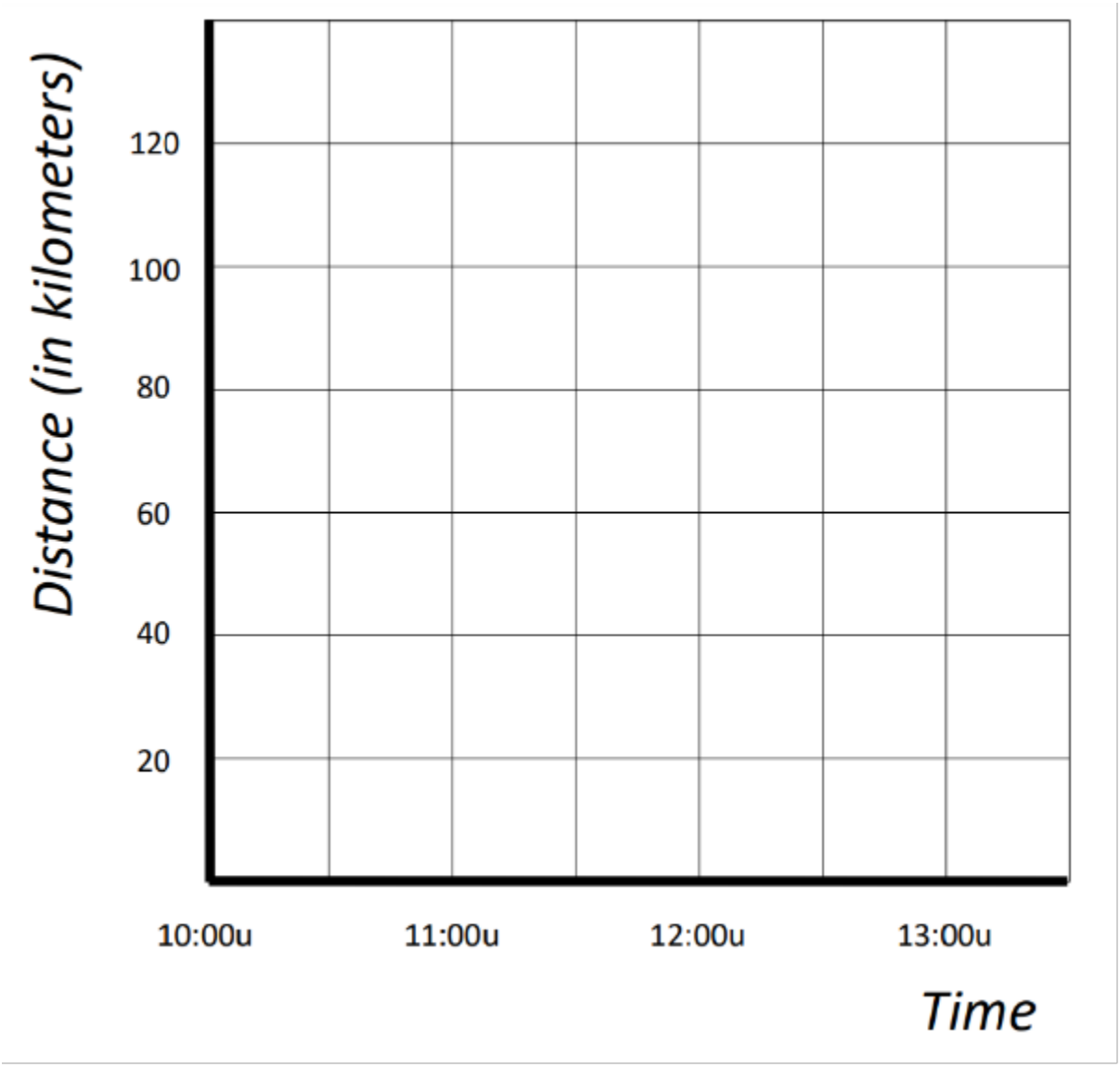


Between which points does the car goes fastest? How do you know?

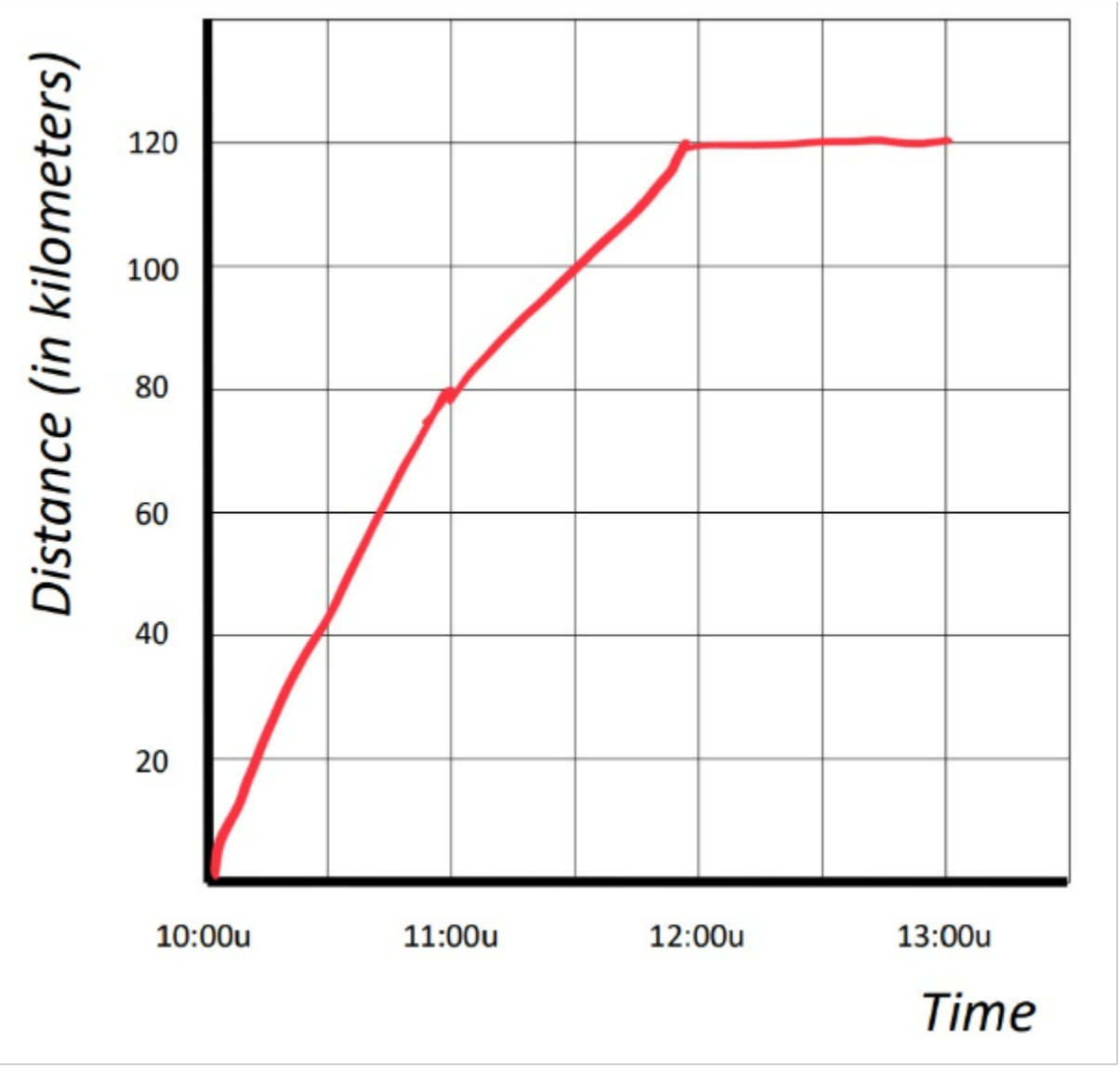
2. **A train ride.**

A train travels **twice as fast** between **10:00 and 11:00** than between **11:00 and 12:00**. The train stands still from **12:00 to 13:00.**

Draw a graph that fits the description above. How do you know?



Right answer:



Scoring

1. **A car drive.**

Levels of reasoning with increasing sophistication:

R0: unrelated reasoning

R1: Iconic reasoning

R2: Single variable reasoning

R3: Multiple variable reasoning

2. **A train ride.**

Levels of reasoning with increasing sophistication (based on the graphical solutions)

R0: an illogical graph without taking into account the description of the motion situation

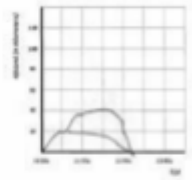
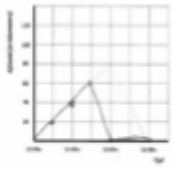
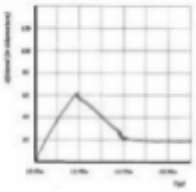
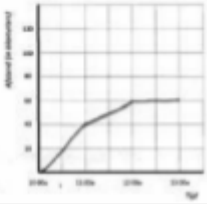
R1: A graph based on superficial characteristics of the motion event

R2: A graph taking into account a single variable correctly

R3: A graph taking into account multiple variables correctly

Table 4

Coding scheme used for students' level of reasoning on the graph interpretation and graph construction tasks

		Description of students' reasoning	
		Graph interpretation	Graph construction
Level of reasoning	Code	*Example*	*Example*
		Student reasons...	**Student constructs graph...**
Unrelated reasoning	R0	...without referring to the graphical representation or the motion event *"You can see"* *"I guessed"*	...without taking the description of the motion event into account 
Iconic reasoning	R1	...on the basis of the shape of the graphical representation or superficial characteristics of the motion event *"Because those two points are the highest"* *"Over there the line is the longest"*	...on the basis of superficial characteristics of the description of the motion event 
Single variable reasoning	R2	...on the basis of a single variable (distance or time or speed) *"Between B and C, the line goes upwards from 4 till 12, so he gives a lot of gas"* *"There he drives 8 kilometers and everywhere else this is 4 or less"*	...taking into consideration a single variable (distance or time or speed) 
Multiple variable reasoning	R3	...on the basis of multiple variables (distance and/or time and/or speed) *"The car drives 8 kilometers in 5 minutes. So, in the shortest period of time, the most kilometers."*	...taking into consideration multiple variables (distance and/or time and/or speed) 

Note. The complete coding scheme, including examples of student responses per task, can be found in Appendix 4.1 (graph interpretation) and Appendix 4.2 (graph construction).