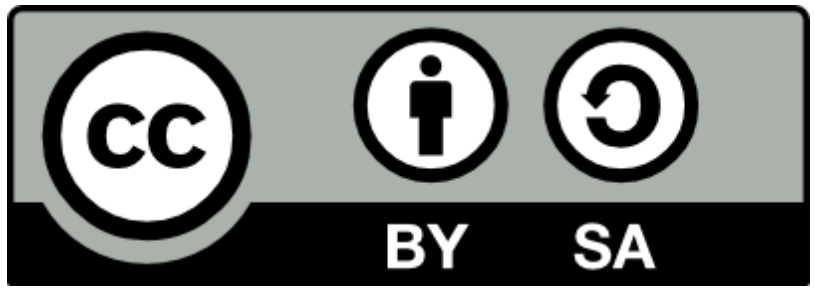

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HOGESCHOOL ipabo

Utrecht University

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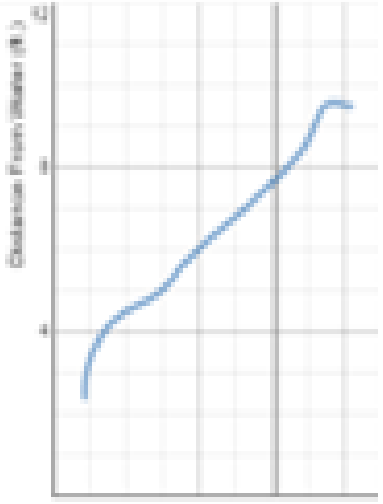
from primary to upper secondary school



Teacher training course (6h) – part 1

Conference, Place

Date

Functions and functional thinking

- Theory of functions
- Recognizing theory in design
- Representations of functions
- Representations in tasks
- Functions in Classroom

Functions and functional thinking

- **Theory of functions**
  - Recognizing theory in design
  - Representations of functions
  - Representations in tasks
  - Functions in Classroom

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Activity A: What functions are these?

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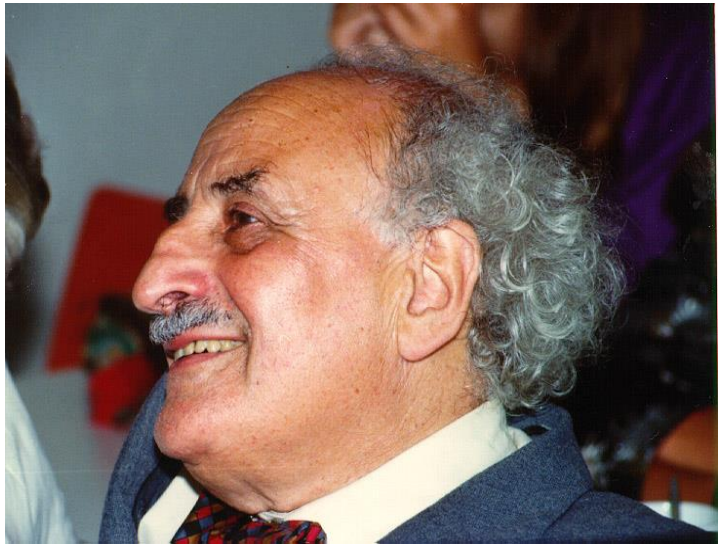
from primary to upper secondary school

The concept of function

Freudenthal (1983, p. 496) about function:

“the directedness from something that varies freely to something that varies under constraint.”



Source: https://www.uu.nl/en/research/freudenthal-institute/about-us/background/mathematics-education

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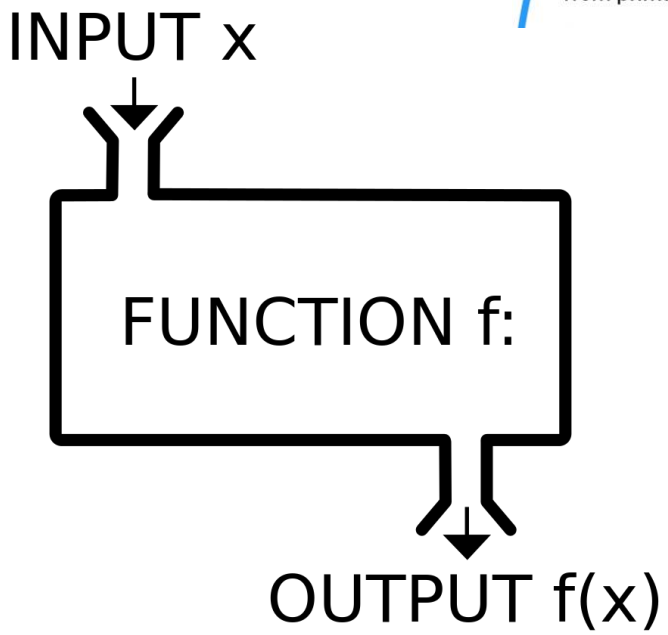
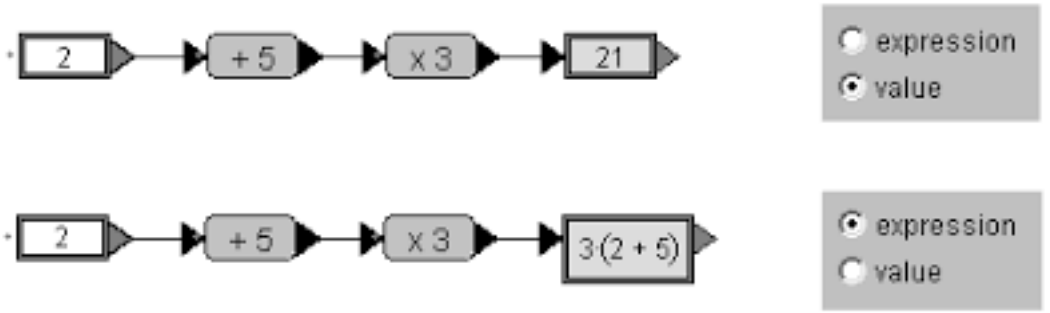
Perspectives on functions

- **Four aspects:**
- **Input-output assignment**
- **Covariation**
- **Correspndence**
- **Mathematical object**





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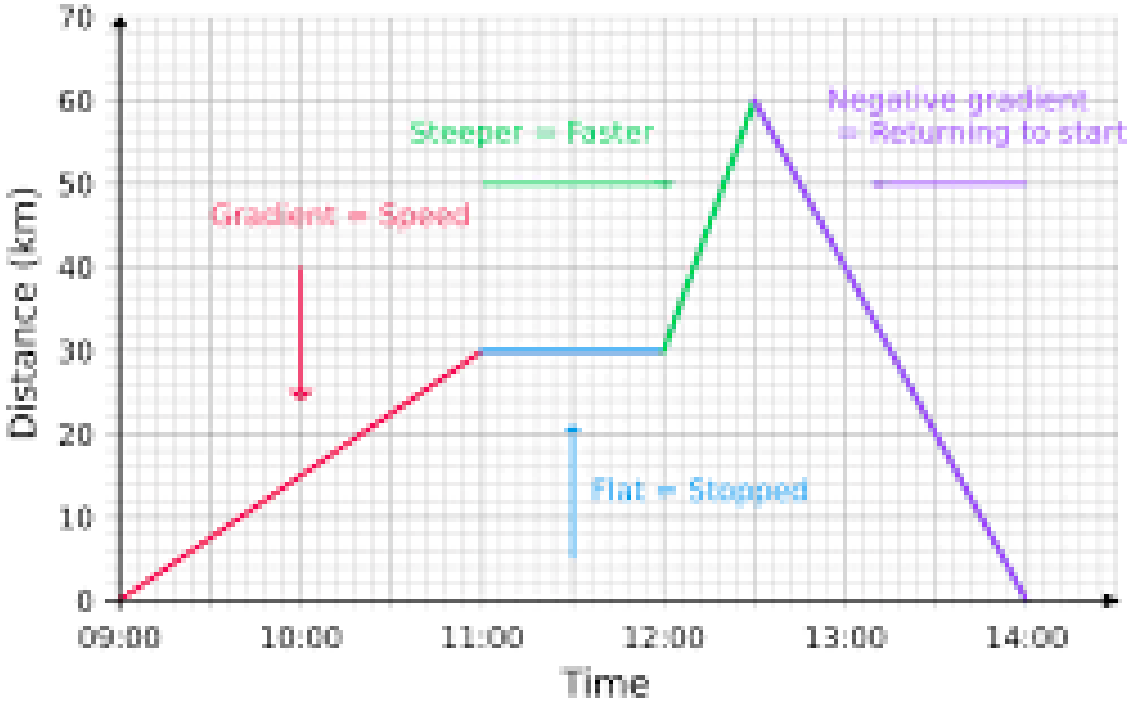
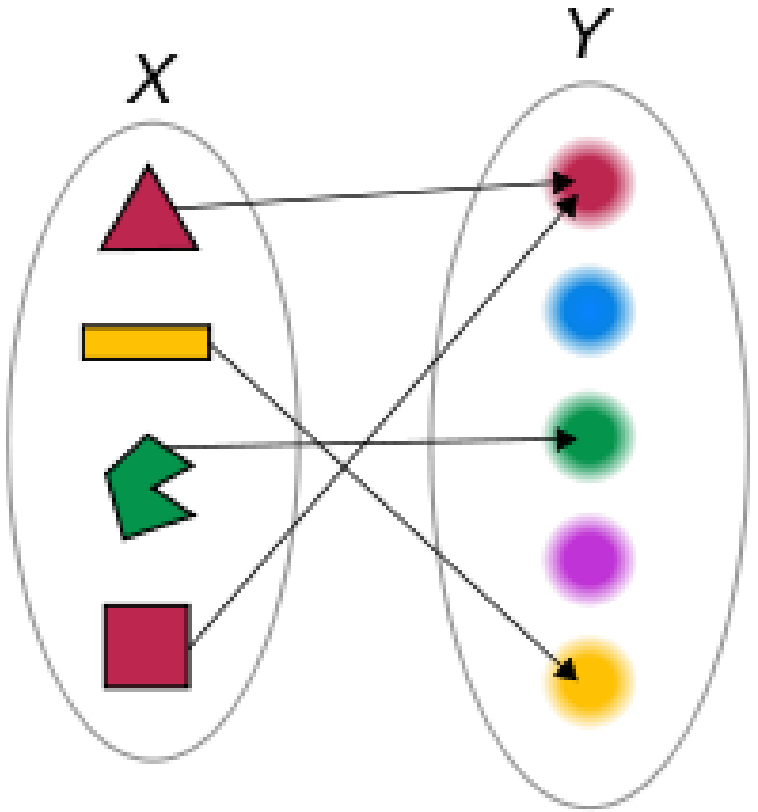
Perspectives on functions

- **Four aspects:**
  - **Input-output assignment**
  - **Covariation**
  - **Correspndence**
  - **Mathematical object**



Perspectives on functions

- **Four aspects:**
- **Input-output assignment**
- **Covariation**
- **Correspndence**
- **Mathematical object**



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Perspectives on functions

- **Four aspects:**
- **Input-output assignment**
- **Covariation**
- **Correspndence**
- **Mathematical object**

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

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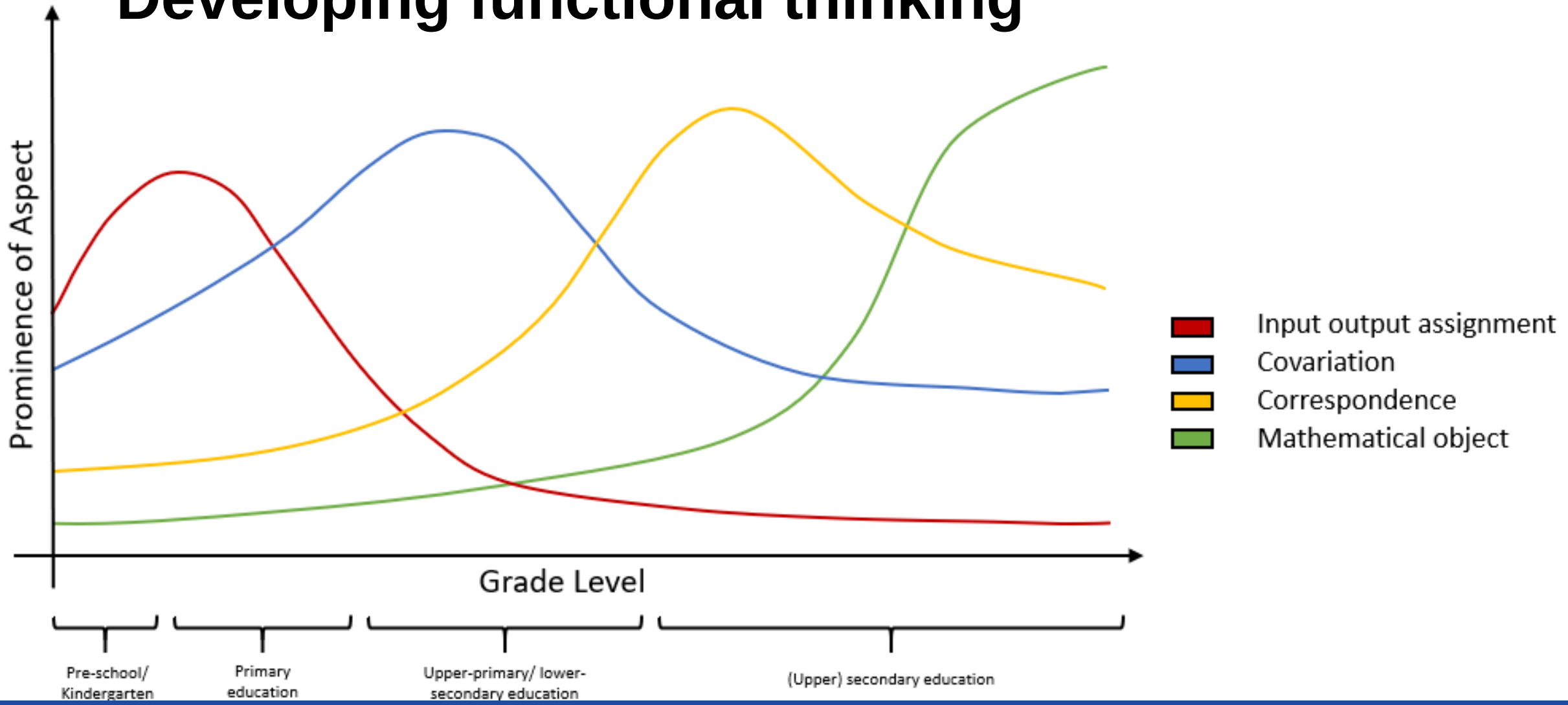
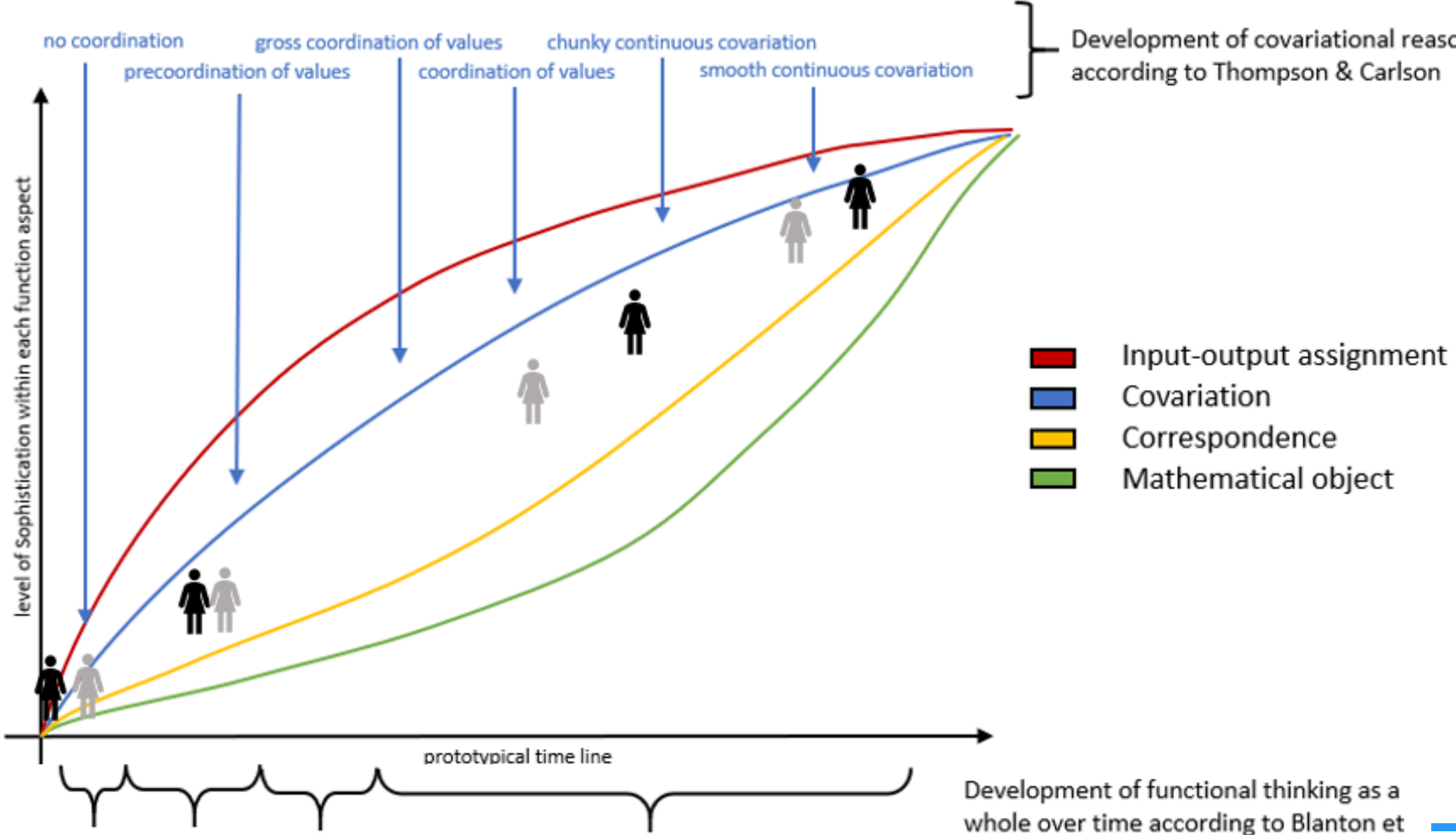
Developing functional thinking




no coordination
precoordination of values
gross coordination of values
coordination of values
chunky continuous covariation
smooth continuous covariation
Development of covariational reasoning according to Thompson & Carlson
level of Sophistication within each function aspect
Input-output assignment
Covariation
Correspondence
Mathematical object
prototypical time line
No functional thinking
variational thinking
covariational thinking
correspondence thinking
Development of functional thinking as a whole over time according to Blanton et al/ Stephens et al

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Functional Thinking…

- **… is thinking in terms of relationships, dependencies and change**
- **… is the process of describing, building and reasoning with and about functions, with the four mentioned perspectives**

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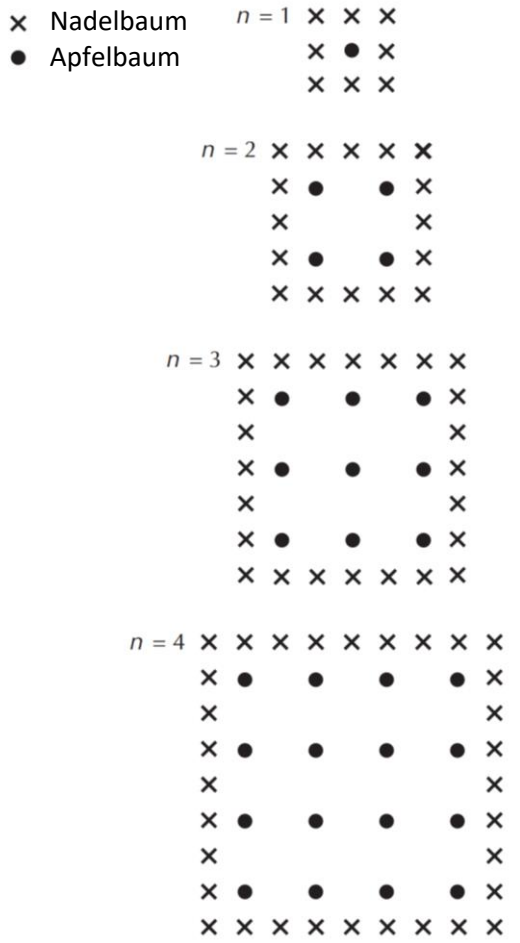
Functional Thinking…

A farmer plants apple trees in a square pattern. To protect the trees from wind, he places conifers all around the orchard.

The situation is shown opposite with the pattern of apple trees and conifers for different numbers (n) of rows of apple trees.

When is the number of conifers the same as the number of apple trees?



(Aufgabe übersetzt von OECD, 2009, S. 102)

- **… is more than just applying procedures**

Functions and functional thinking

- Theory of functions
- **Recognizing theory in design**
- Representations of functions
- Representations in tasks
- Functions in Classroom

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“embodied design” for function

The idea of embodiment:

- **All knowledge/ cognition is rooted in physical experience, in alternation of action and perception.**
- **“mathematics in the fingers” – new ICT tools (e.g. GeoGebra) provide opportunities for this**

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Activity B: Recognizing theory in design

Go through a digital environment (nomograms). For each of the four aspects of function, find an item that is ideally situated to it. Write down!

10-15 minutes
In pairs
Afterwards: class exchange.

- **Four aspects:**
  - **Input-output assignment**
  - **Covariation**
  - **Correspndence**
  - **Mathematical object**

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Wrap up

- **Four aspects:**
  - **Input-output assignment**
  - **Covariation**
  - **Correspndence**
  - **Mathematical object**
- **Item:**
  - **…**
  - **…**
  - **…**
  - **…**

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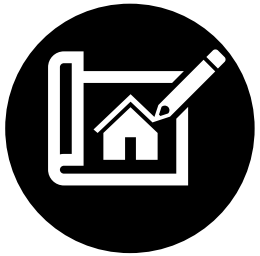
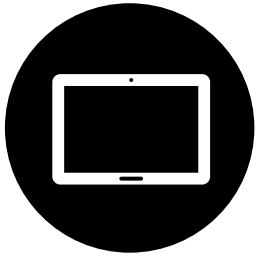
Design principles for functions

INQUIRY BASED LEARNING AND TEACHING



SITUATEDNESS

EMBODIMENT



(DIGITAL) TOOLS

Functions and functional thinking

- Theory of functions
- Recognizing theory in design
- **Representations of functions**
- Representations in tasks
- Functions in Classroom

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Representation of functions

Representation is a crucial element for a theory of mathematics teaching and learning, not only because the use of symbolic systems is so important in mathematics, the syntax and semantics of which are rich, varied, and universal but also for two strong reasons: (a) mathematics plays an essential part in conceptualizing the real world, (b) mathematics makes a wide use of homomorphism in which the reduction of structures to another is essential (Vergnaud, 1987, p.227)

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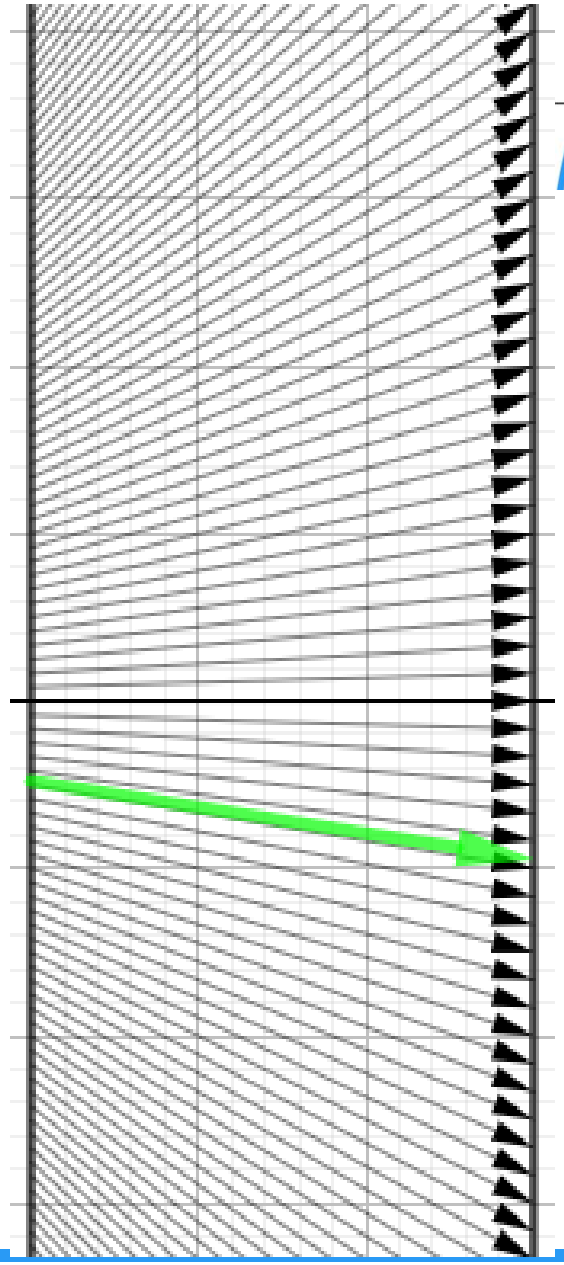
Representation of functions

- Natural language
- Formulas
- Tables
- Nomograms
- Charts
- Chains of machines
- …



Functions and functional thinking

- Theory of functions
- Recognizing theory in design
- Representations of functions
- **Representations in tasks**
- Functions in Classroom

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Activity C:

Find/ create one or more problems for undergraduate students that are solved using functional thinking, but in a different presentation from the one in which it is posed.

10-15 minutes
In pairs or in groupes of three
At the whiteboards
Afterwards: presentation to each other

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from primary to upper secondary school

Activites during representational change

	Situations, Verbal Description	Tables	Graphs	Formulae
Situations, Verbal Description	Rephrasing, Simplifying the real situation	Measuring	Sketching; Visualizing a situation	Modelling
Tables	Reading	Refining or Coarsening the table	Plotting	Fitting
Graphs	Interpretation	Reading Off	Changing the axis scaling	Curve Fitting
Formulae	Parameter Recognition	Computing	Sketching	Algebraic transformation

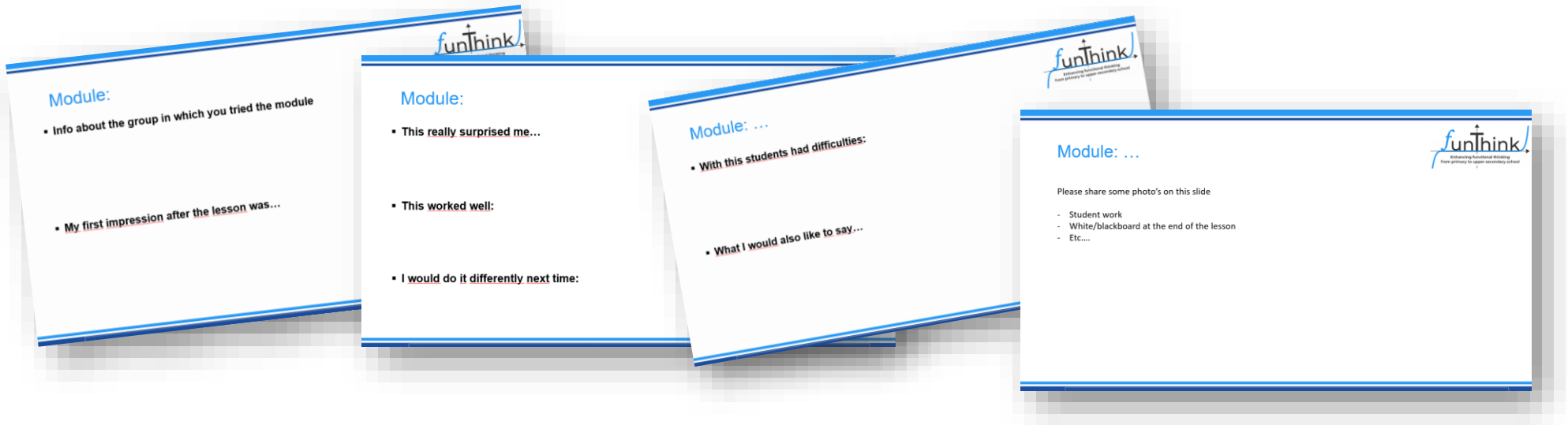
(Barzel et al., 2021, p. 75; Janvier, 1978, p.97)

Functions and functional thinking

- Theory of functions
- Recognizing theory in design
- Representations of functions
- Representations in tasks
- **Functions in Classroom**

How do we continue to work (in classroom)?

Distance phase/ Homework: Trying out a module/ task in your own classroom.



To do this, use the slides „Report_classroom_experience_with_FunThink_module-1“

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Thank you for your attention!

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Module:

- **Info about the group in which you tried the module**

- **My first impression after the lesson was…**

Module:

- **This really surprised me…**
- **This worked well:**
- **I would do it differently next time:**

Module: …

- **With this students had difficulties:**

- **What I would also like to say…**

Module: …

Please share some photo’s on this slide

- Student work
- White/blackboard at the end of the lesson
- Etc….

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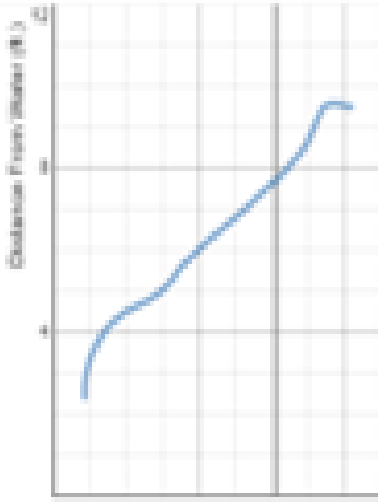

Teacher training course (6h) – part 2

Conference, Place

Date

Functions in classroom

- Exchange of teaching experience
- Reflecting theory in design
- Strategies and difficulties of students
- (Re)design your own acitivity

Functions in classroom

- **Exchange of teaching experience**
- **Reflecting theory in design**
- **Strategies and difficulties of students**
- **(Re)design your own acitivity**

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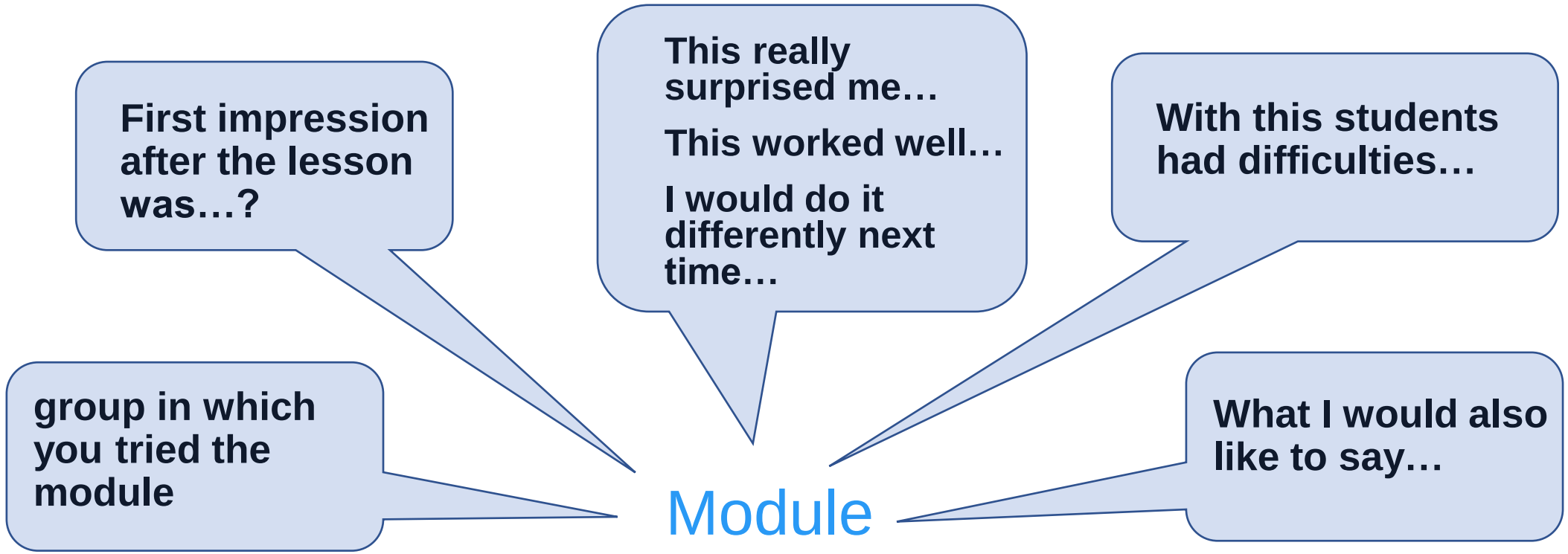
Distance phase/ Homework: Trying out a module/ task in your own classroom.



Functions in classroom

- Exchange of teaching experience
- **Reflecting theory in design**
- Strategies and difficulties of students
- (Re)design your own acitivity

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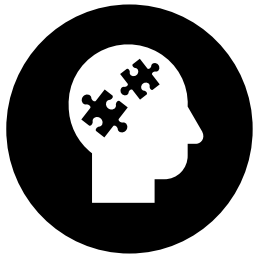
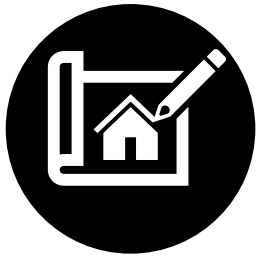
Design principles for functions

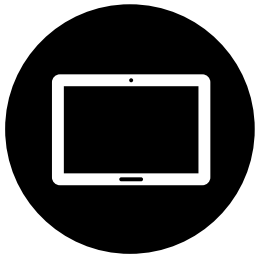


INQUIRY BASED LEARNING AND TEACHING



SITUATEDNESS

EMBODIMENT



(DIGITAL) TOOLS

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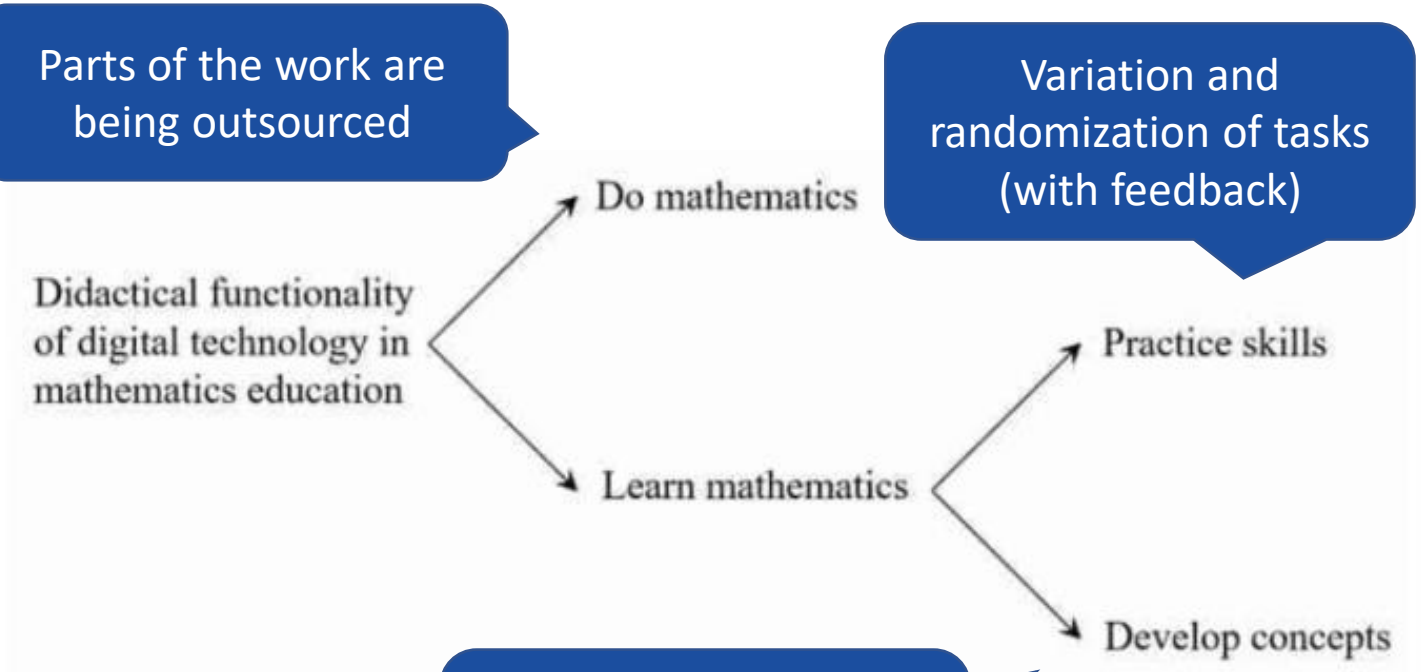
Tool use: dimensions

- **Limited and/or specific functionality vs general-purpose tools**
- **Mathematical domains**

closely related

- **Didactical functionality**
  - Type of task
  - Teaching and learning process



→ **Important for the decision what tool to use when**

Drijvers, 2019

Reflecting theory in design

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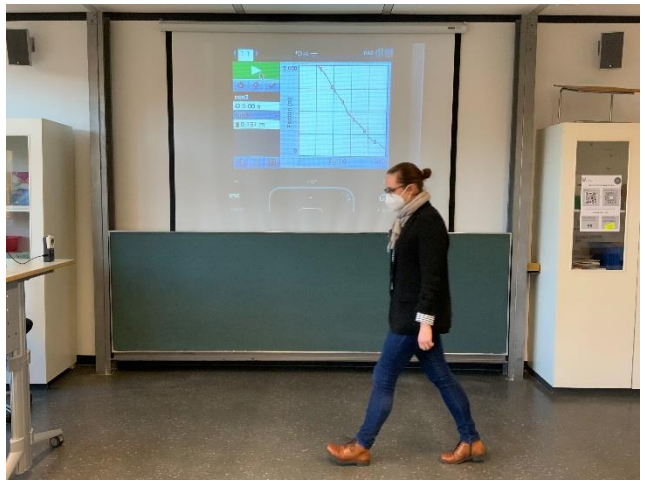
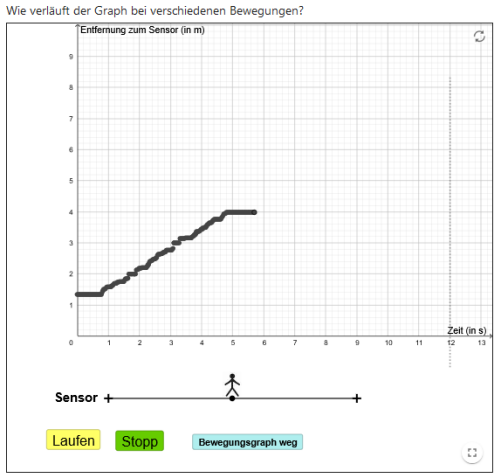
How do these dimensions (general functionality, mathematical domain, didactical functionality) apply to the tool used in the learning environment “Walking the graph”?





•https://www.geogebra.org/m/rxtznqvc

Sensor and program
→ sensor & program: limited and specific function, development of concepts

GeoGebra:
→ here: specific functionality in one domain for the development of a concept

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Experiments with real and digital tools to foster functional thinking

Using real and digital tools, e.g., to foster functional thinking by experiments

Key findings of literature:

- Digital simulations should complement experiments with real materials
  - Promotion of correspondence view through real experiment
  - Promotion of covariational view especially through qualitative simulation
  - Discourse on covariation essential for learning growth
  - Covariational view accessible even to learners at low skill levels
- Digital learning environments should be embedded in paper-pencil environment
  - Taking notes / Logging supports reflection
  - Better availability of the paper-pencil protocol

Best effects: Combination of both approaches
(Digel et al., 2023)

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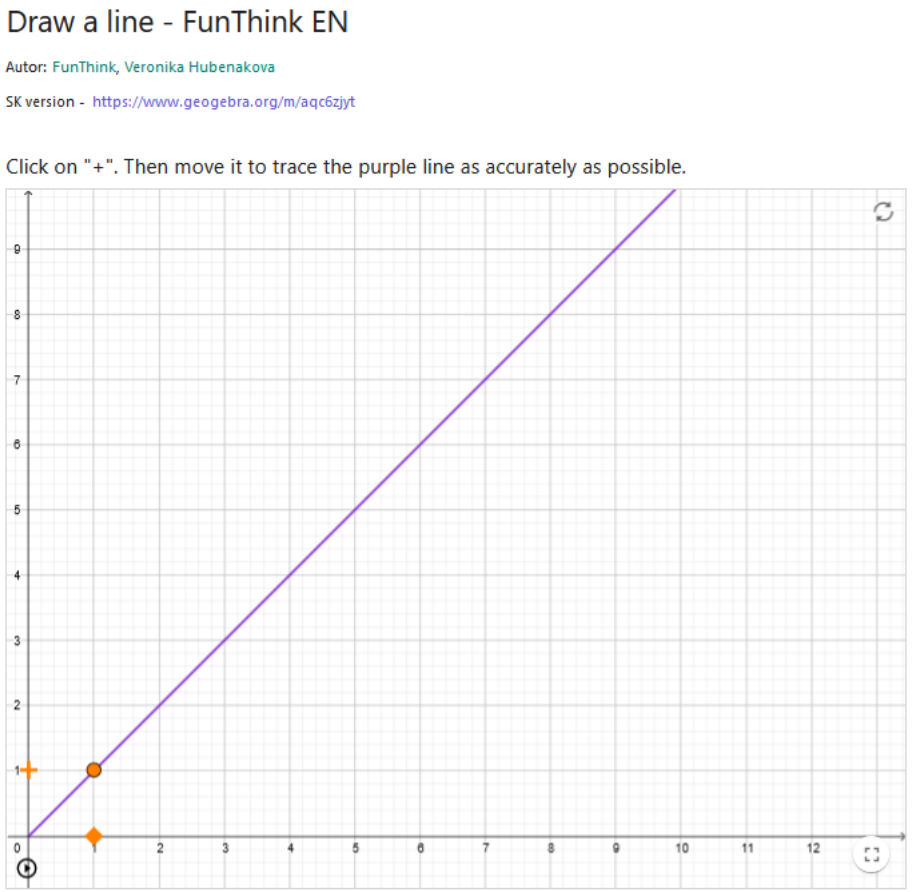
Activity B: Analyze a task

Analyze a task (maybe the one tried out in classroom) with respect to the design ciriteria. What ist the added value of the tool? Does tool help to develop functional thinking? In what way? Could that be improved?

10-15 minutes
In pairs or in groupes of three



https://t1p.de/e3w47

Functions in classroom

- Exchange of teaching experience
- Reflecting theory in design
- **Strategies and difficulties of students**
- (Re)design your own acitivity

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from primary to upper secondary school

Activity C: Watching a video from classroom learning episode

Watch the video. You will see 2 students drawing their own graph.

Keep the following questions in mind and discuss in small groups:

- How would you describe the student answers for student 1 and 2?
- Do you recognize any misconceptions among the students?
- What instruction could be helpful to support the students‘functional thinking?
- What else do you notice?

10 minutes

Activity C: Watching a video from classroom learning episode

Watch implementation video walking graph sexploration. Available here: https://www.funthink.eu/learning-environments/lower-secondary-education/walking-graphs (after login)

Strategies and difficulties of students

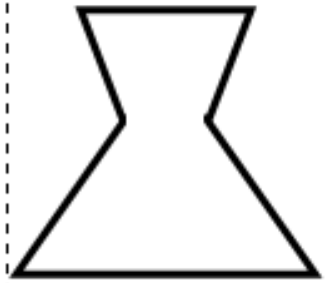
Activites during representational change

	Situations, Verbal Description	Tables	Graphs	Formulae
Situations, Verbal Description	Rephrasing, Simplifying the real situation	Measuring	Sketching; Visualizing a situation	Modelling
Tables	Reading	Refining or Coarsening the table	Plotting	Fitting
Graphs	**Interpretation**	Reading Off	Changing the axis scaling	Curve Fitting
Formulae	Parameter Recognition	Computing	Sketching	Algebraic transformation

Some Strategies and difficulties of students

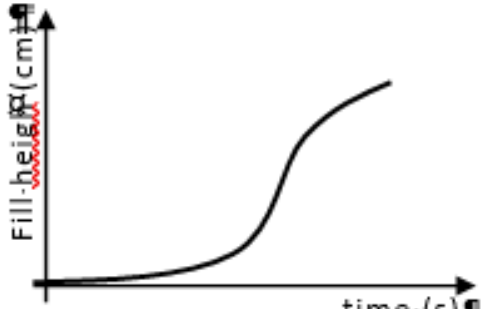
(Barzel et al., 2021, S. 75; Janvier, 1978)

Task



A flower vessel as displayed on the left is filled by a uniform water jet.

Which of the graphs represents best the relationship between time and fill height?



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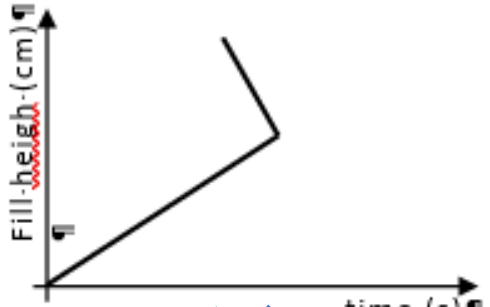

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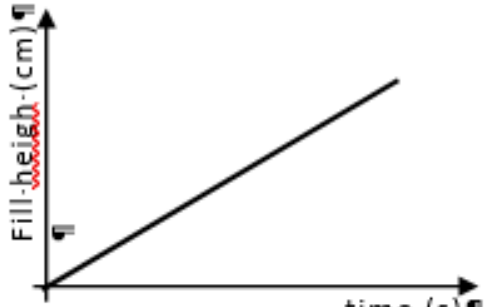

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from primary to upper secondary school

		table	Graphs	Formulae
Graphs	**Interpretation**	Reading Off	Sketching; Visualizing a situation	Modelling
			Plotting	Fitting
Graphs		Reading Off	Changing the axis scaling	Curve Fitting
Formulae	Parameter Recognition	Computing	Sketching	Algebraic transformation

Some Strategies and difficulties of students

(Barzel et al., 2021, S. 75; Janvier, 1978)

Strategies and difficulties of students

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Activites during representational change

	Situations, Verbal Description	Tables	Graphs	Formulae
Situations, Verbal Description	Rephrasing, Simplifying the real situation	Measuring	**Sketching, Visualizing a situation**	Modelling
Tables	Reading	Refining or Coarsening the table	Plotting	Fitting
Graphs	Interpretation	Reading Off	Changing the axis scaling	Curve Fitting
Formulae	Parameter Recognition	Computing	Sketching	Algebraic transformation

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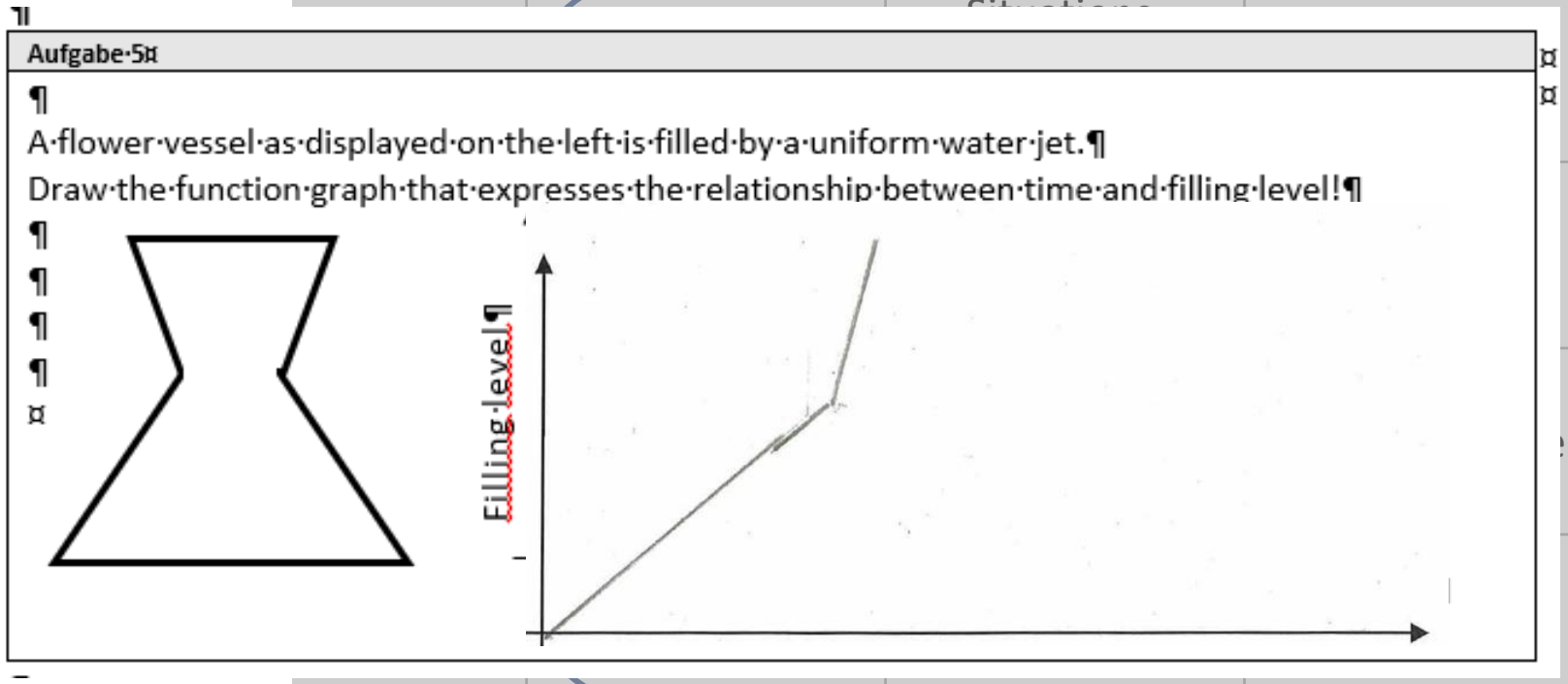
Activites during representational change



			Graphs	Formulae
			Sketching, Visualizing a situation	Modelling
			Plotting	Fitting
			Changing the axis scaling	Curve Fitting
Formulae	Parameter Recognition	Computing	Sketching	Algebraic transformation

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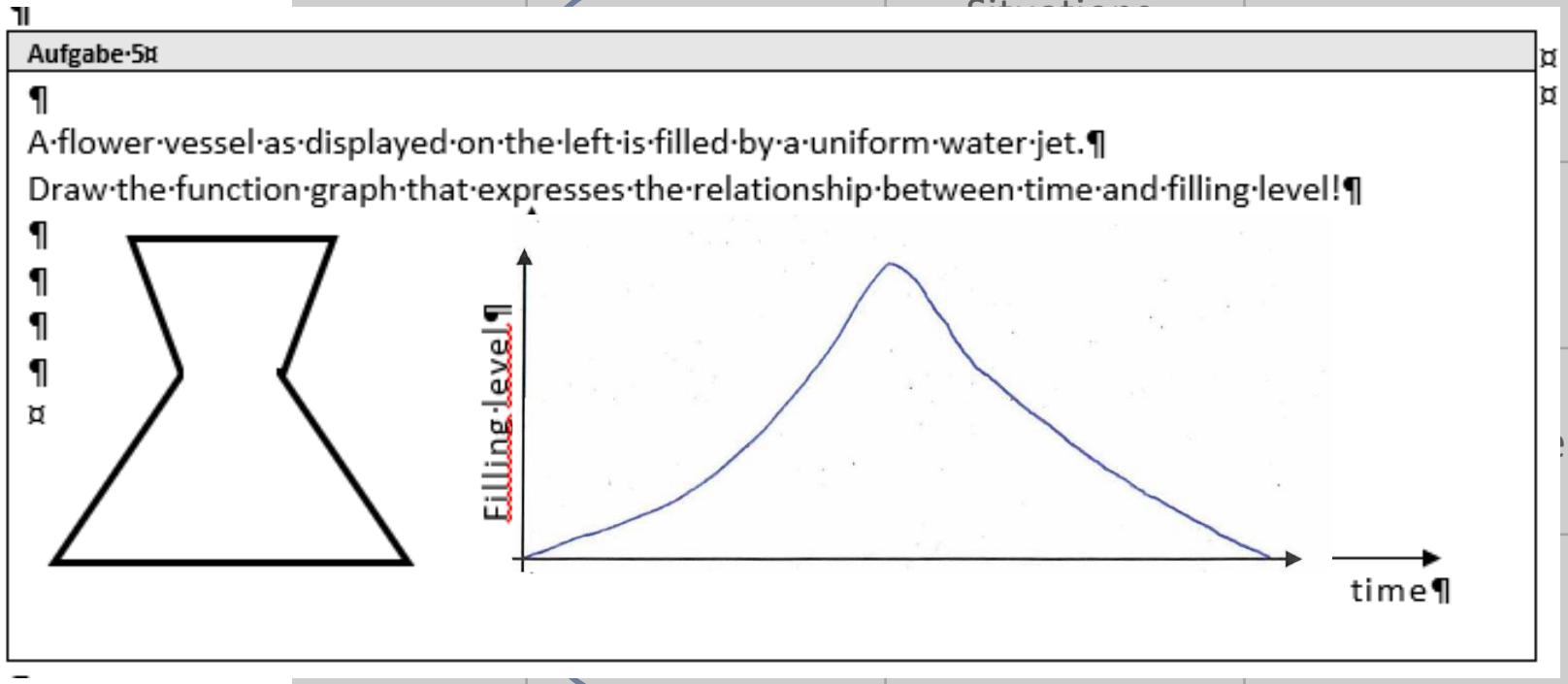

			Graphs	Formulae
			Sketching, Visualizing a situation	Modelling
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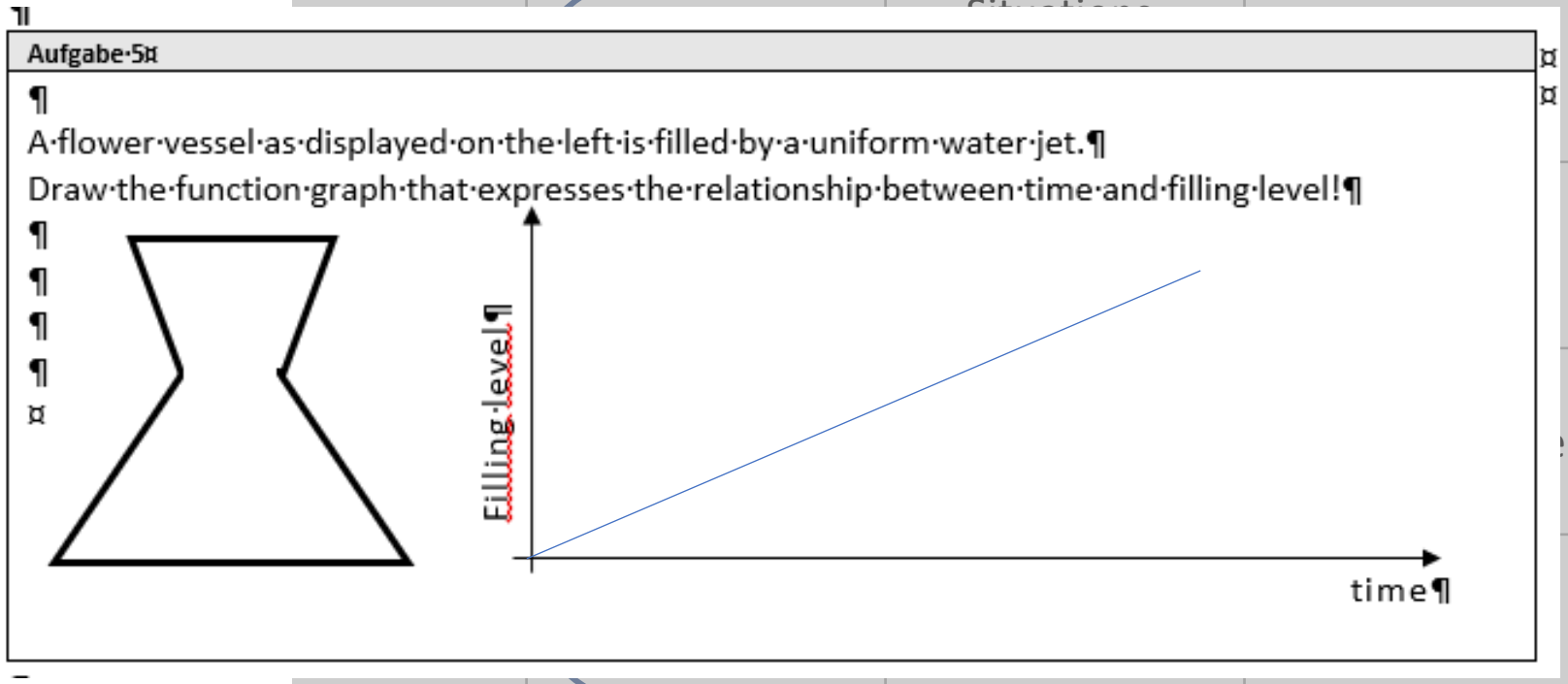
Activites during representational change



			Graphs	Formulae
			Sketching, Visualizing a situation	Modelling
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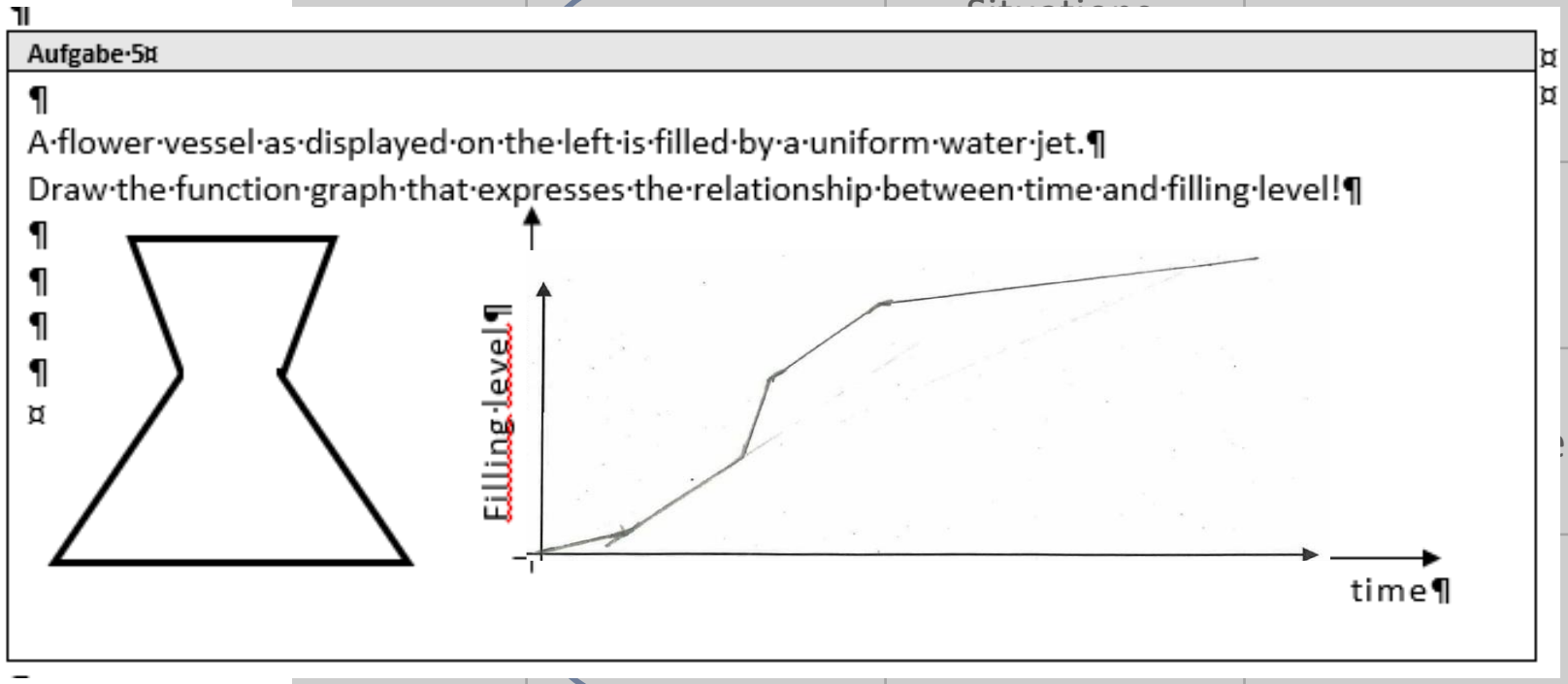

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Some Strategies and difficulties of students

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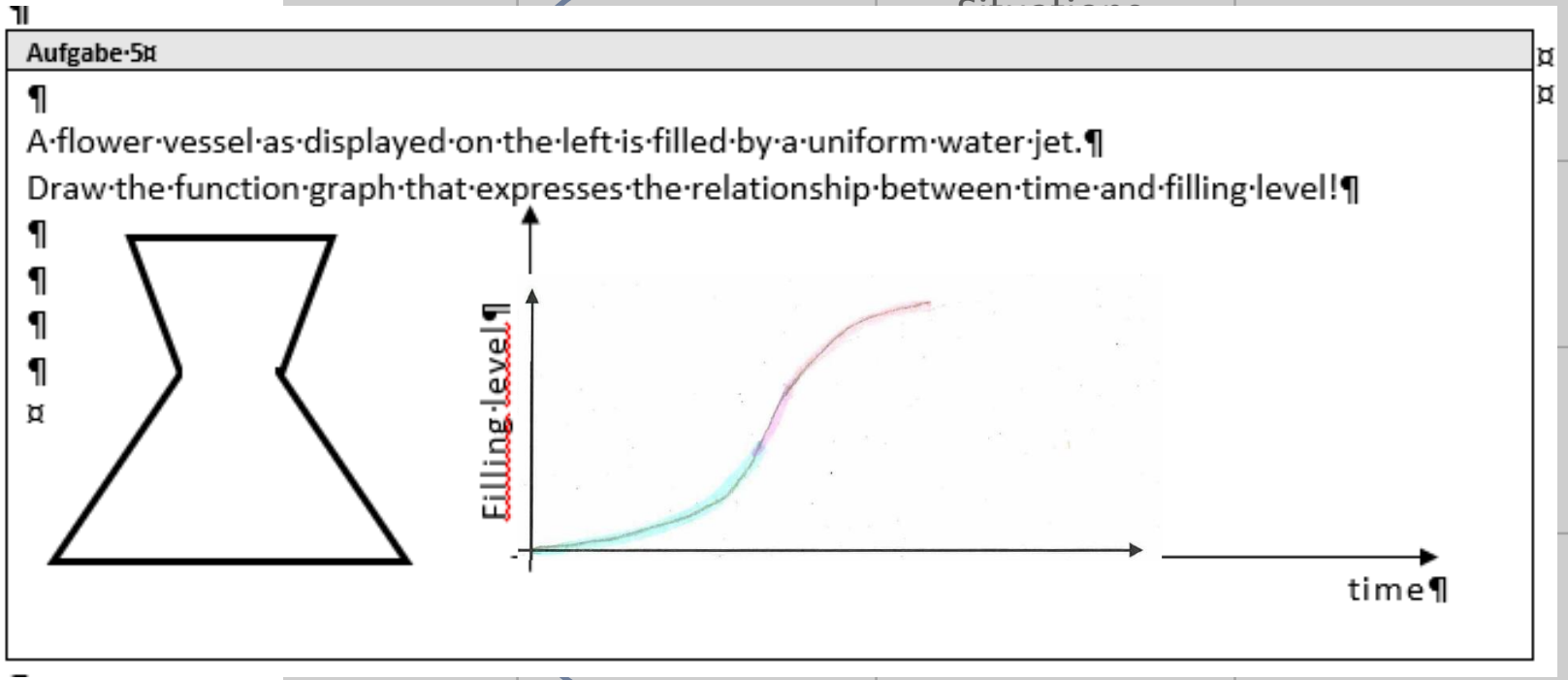
Activites during representational change



			Graphs	Formulae
			Sketching, Visualizing a situation	Modelling
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Some Strategies and difficulties of students

(Barzel et al., 2021, S. 75; Janvier, 1978)

Functions in classroom

- Exchange of teaching experience
- Reflecting theory in design
- Strategies and difficulties of students
- **(Re)design your own acitivity**

Activity D: (Re)design your own activity

Look at task in different math books.

1. Find three tasks that promote functional thinking.Which design principles are integrated?
2. Identify at least one task that needs improvement to promote functional thinking.
3. Modify the task that needs improvement so that functional thinking can be optimally promoted. Integrate the design principles.

20-25 minutes

In pairs or in groupes of three

Afterwards: presentation to each other

Learning goals…

You are able

- to formulate in your own words what functional thinking entails.
- to identify the different aspects of functional thinking in learning environments
- to find opportunities for its developement in (lower) secondary school
- to analyze misconceptions, difficulties and successes as well as chosen strategies
- to recognize differences between students in the learning process of functional thinking
- …

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Thank you for your attention!

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