Vision document on Functional Thinking

Version 2.0 – November 5, 2021

Pre-amble

In this Erasmus+ FunThink vision document, we aim to refine our view on what functional thinking (FT) is and on how it can be learned and taught, as to inform the design of learning environments. We want to share our view on functional thinking with the reader, being a teacher, a teacher educator, a researcher, or another stakeholder in the field of mathematics education.

Based on the state-of-the-art literature on functional thinking, interviews with educators, and an inventory of the national curricula in the participating countries, this vision document first elaborates the notion of functional thinking. Next, we describe possible phases in the development of functional thinking throughout primary and secondary education. This leads to the identification of design guidelines for learning environments focusing on functional thinking. The vision document finishes with more detailed information on the literature study, the national curricula, and the interviews with educators.

This vision document will guide the project's next steps. Based on the views and guidelines expressed here, we will develop innovative teaching-learning environments that aim at fostering students' functional thinking in primary and secondary education. These learning environments will include embodied and inquiry-based approaches and make use of up-to-date digital technologies. Additional teacher guides will explain the design rationale and provide suggestions for effective implementation in practice.

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1. Functional thinking

In mathematics education worldwide, functional thinking clearly is “in the air”. A search for this expression in Google Scholar provides 5780 hits (April 2, 2021), more than half of them stemming from the last decade. This attention for functional thinking might fit in an overall growing interest in mathematical thinking, such as algebraic thinking, computational thinking, algorithmic thinking, and in different kinds of literacy, such as mathematical literacy, scientific literacy, statistical literacy and digital literacy. What these notions share, is a focus on higher order knowledge and skills; in the meantime, they are somewhat difficult to define and to demarcate. Therefore, our aim in this section is to pinpoint the notion of functional thinking, as to develop a shared view. To do so, we start out with a societal perspective, followed by a mathematics education stance and a mathematical perspective.

Functional thinking, considered a way of thinking in terms of relationships, interdependencies, and change, is crucial in society. Not only when working on various problems in mathematics and other school subjects (e.g., natural sciences, geography, social studies), but also future education (e.g., in sciences, economy, medicine, or psychology) and professional and everyday life, people may benefit from the ability to think in terms of causal relationships, associations between variables, and quantifiable dependency. Examples for such use of functional thinking include understanding scientific laws, monitoring the redemption of bank credit, or understanding virus spread models such as the current COVID-19 in terms of exponential growth. The relevance of functional thinking in private, academic, and professional contexts implies that fostering functional thinking in education is vital.

Therefore, let us take an educational stance. Interestingly, many of the mathematics educators who took part in the project’s interview study initially understand the term ‘functional’ as related to the way mathematics is used, to the way it functions in other contexts such as science and life: functional in the sense of how one can use it. Of course, we acknowledge mathematics being functional in solving problems in a whole range of problem situations and application contexts. However, with functional thinking we restrict ourselves to these types of applications through using the notion of mathematical function. The importance of this concept is reflected in the central role it plays in mathematics education, and in mathematics curricula in different countries (see Section 5). Already in primary education, early algebra activities include attention to pattern recognition and the study of input-output relationships. In secondary education, much teaching time is devoted to the investigation of specific classes of functions, e.g., linear, quadratic, power, exponential, and trigonometric functions, including different representations and different algebraic techniques to deal with them. Functions are used to model phenomena from other mathematical domains such as geometry, from scientific contexts, and from situations outside the world of science and mathematics. Dealing with mathematical functions, however, encompasses more than the ability to manipulate the formulas representing them: it involves dealing with the notion of function in its versatility, and developing a rich concept image, that includes aspects such as representation, generalization, causality, regularity, and covariation. To acknowledge this versatility, which is not easy to capture, the notion of functional thinking emerged.

Functional thinking is defined in different ways. As a starting point, we follow the often-used description of functional thinking as the process of building, describing, and reasoning with and about functions (Blanton et al., 2015; Pittalis et al., 2020). Other descriptions have a more specific focus. For example, Smith (2008) and Markworth (2012) highlight the representation and generalization aspect of functional thinking, which is described as a type of

> […] representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances (p. 143)

In a somewhat wider interpretation, functional thinking connects to mathematical notions of structure, (co-)variation, change, and relation. It also concerns the ideas of qualitative change, quantitative change, relationships between these changes, and using these relationships to solve problems (Cañadas et al., 2016).

Unfortunately, the learning and teaching of functional thinking is far from straightforward. Difficulties lie in the abstract character of functions that are only accessible via specific representations (e. g., graph, equation, table, situation description), and in the need to go back-and-forth between mathematics and real-world contexts in processes of modeling and horizontal mathematization (see Appendix B for an illustration of types of mathematization). Promising didactical approaches to functional thinking education include connections to real-world situations, embodied activities, the use of digital technology, and an inquiry-based approach.

To better understand the educational challenges, let us consider functional thinking from a mathematical perspective. The concept of function, at the heart of functional thinking, is fundamental, and has had a long and difficult history of development, in which different views on function have emerged. These different aspects highlight the key characteristics of mathematical function and can inform the teaching and learning of functional thinking. In line with other literature (e.g., Doorman et al., 2012), we follow Pittalis et al. (2020) in distinguishing the following main views on function and functional thinking:

a. *The input-output assignment view*
This view on function as an input-output machine stresses the operational and computational character of the function concept. It includes exploring how a particular input value will lead to an output value. Questions on the rules that determine the output based in the input will naturally arise. As an example, one can consider the total amount to pay as a function of the number of objects (candies, tickets) bought. Suitable function representations are the input-calculation-output arrow chain or the input-output table.

b. *The dynamic process of covariation view*
This view emphasizes the covariation of the dependent variable with the independent variable. It relates to the work by Thompson and Carlson (2017) on covariational reasoning. A function is seen as “two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person’s conception, every value of one quantity determines exactly one value of the other” (p. 436). This view highlights that the independent variable, while running through the source domain set, causes the dependent variable to run through a range set. One may question how one variable will change while the other one varies, and how the process of dynamic covariation works. As an example, one may study how distance traveled changes as a function of time. Suitable representations may be the function value table or the function graph, which can be scrolled through or traced.

c. *The correspondence relation view*
This view on function concerns understanding the relation between the independent and the dependent variable and being able to represent it. This includes the mapping view and may lead to the more formal definition of function as a set of ordered pairs. This view helps to answer

questions on the global character of the relationship. As an example, one may think of the types of correspondence between different phenomena, such as age and COVID risks. Suitable presentations may be function graphs and nomograms.

d. *The mathematical object view*
In this view, a function is a mathematical object, with its own representations and properties, that can be compared to other mathematical objects or functions. It is seen as a member of a family of functions, and can be submitted to higher-order processes, such as differentiation. Questions may concern function family properties, and similarities and differences between various families of functions. As an example, one can study the family of polynomials and the family of exponential functions and identify different characteristics. Suitable representations may include (sheaves of) graphs or symbolic representations.

This set of four views has some taxonomy characteristics, in the sense that it may suggest an order in which to acquire functional thinking. Also, we notice a gradual development from a process view (function as an input-output process) to a more structural view (function as a mathematical object; cf. Sfard, 1991), and from a more local, pointwise perspective to a global view. This being said, of course the different views are intertwined; it is not always straightforward to disentangle them within student reasoning. Still, the four aspects offer a framework to design tasks and to study student work in the domain of functional thinking.

To summarize, we see functional thinking as the process of building, describing, and reasoning with and about functions. It is considered key in mathematical thinking, and relevant for private, academic and professional life. It relates to different mathematical domains, such as algebra, calculus, and geometry, and has applications in a wide range of problem situations. Its basics lie in the four different 'faces' of the mathematical function concept. An important question is how students can acquire functional thinking, which is the topic of the next section.

2. Development of functional thinking

As outlined above, functional thinking involves different views that can be generally characterized by an increasing level of sophistication. Functions in the sense of input-output assignment, a covariational view of functions, and a view of functions as correspondence relation are already accessible for young children at primary and lower secondary school level (Leinhardt et al., 1990; Lichti & Roth, 2019; Pittalis et al., 2020; Stephens et al., 2017). For developing the so-called object view, students need prior experiences with functions to, for example, compare different function classes, operate with, or concatenate functions and apply higher-order processes, such as differentiation or integration. Such a shift from a process-oriented to an object view is described as typical for mathematical conceptualization in general (Dubinsky & Harel, 1992; Sfard, 1991) and can obviously also be referred to the development of the concept of a function (in Ruchniewicz, in press). This exemplarily described progression of sophistication can also be found reflected in numerous curricula such as those of the project partners (see Chapter 5).

In the following, and in line with Clements and Sarama (2014, p. 14) , we understand *levels of sophistication* as "benchmarks of complex growth that represent distinct ways of thinking." Before presenting empirically validated frameworks and theoretical considerations related to levels of sophistication for functional thinking, we want to clarify several issues:

- In line with other researchers (e.g., Blanton, Brizuela, et al., 2015; Stephens et al., 2017), we are convinced that levels of sophistication in children's thinking are directly related to the instructional and curricular framing. Hence, the development of functional thinking cannot be considered separately from the offered learning opportunities. We will come back to this point in Chapter 5, where the partners' national curricula will be presented with regard to particular hinge points in the development of functional thinking.
- Moreover, the level of sophistication that students exhibit with regard to functional thinking might depend on particular task characteristics and hence cannot be considered as a linear and unidirectional path (see, e.g., Blanton, Brizuela, et al., 2015; Pittalis et al., 2020; Stephens et al., 2017; Wilkie, 2014). In this sense, levels can be skipped, students can revert to a presumably lower level of sophistication or work on different levels depending on the context.
- An increase in sophistication in the form of a progression of levels can be based on a theoretical disciplinary perspective, but it can also be informed by empirical data unveiling shifts in student thinking over time (e.g., Battista, 2004; Blanton, Stephens, et al., 2015; Stephens et al., 2017).

In the following, we present empirical findings indicating such a progression of levels and combine them in a graphical representation. As stated in the last two points, these findings always have to be interpreted in the context of the instructional framing that might be influenced by curricular requirements and particular didactical considerations. We will start with more general findings, include insights related to the development of covariational reasoning and end with exemplary topics of all partner countries' curricula supporting the development of functional thinking.

Based on prior work of the group of Maria Blanton (e.g., Blanton, Brizuela, et al., 2015), Stephens and colleagues (2017) present a framework of levels of sophistication that was empirically validated among students from Grade 3 to 5. The eleven levels of this framework will be presented in the following on the basis of the information given in Stephens et al. (2017, pp. 151) with regard to the task displayed in Figure 1 (all examples are from ibid, p. 153):

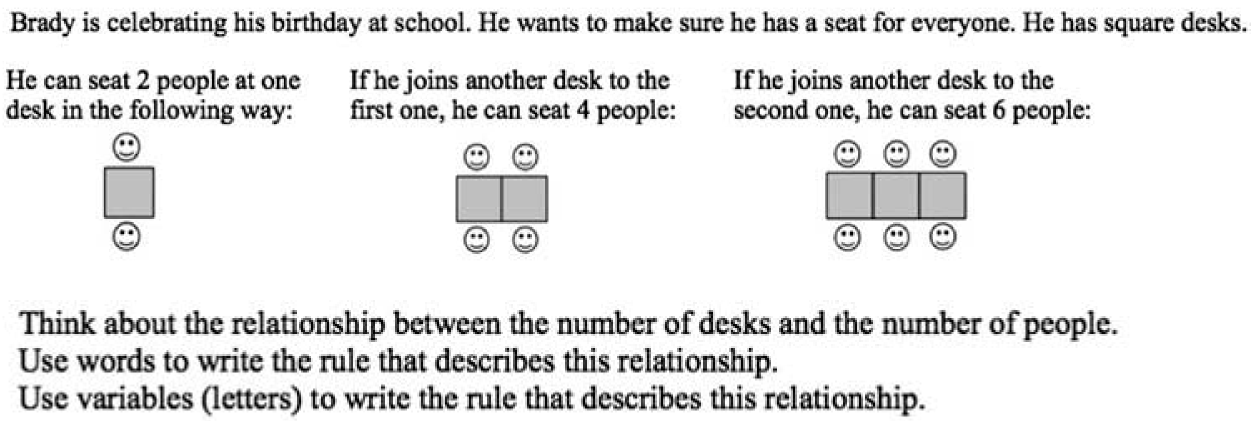


Figure 1: Sample task for explaining the different levels of sophistication (retrieved from Stephens et al., 2017, p. 151; subtasks omitted by the authors)

- L0 (called *prestructural* in Blanton, Brizuela, et al., 2015, p. 525) : Students show no evidence of functional thinking when a corresponding task is presented and do not recognize how the involved mathematical quantities could be related or how this relationship could be expressed; instead, they might describe non-mathematical features of the presented tasks only. For example: "Two people can sit at a table."

- L1 (called *recursive pattern-particular*): Students identify for one or both involved variables a recursive pattern as a sequence of concrete instances. This means that they do not recognize or formulate a general recursive pattern but can describe it on the base of particular numbers. For example: “It goes 2, 4, 6, 8, ….”
- L2 (called *recursive pattern-general*): Students identify for one or both involved variables a recursive pattern without referring only to concrete instances. This means that they are able to describe the pattern in a general way. For example: “The number of people goes up by 2 each time.”
- L3 (called *covariational thinking*): Whereas L1 and L2 are characterized as “variational thinking” without connecting the two variables involved, L3 closes this gap by coordinating them. For example: “Every time you add a desk, you add two more people.”
- L4 (called *single instantiation*): This level is the first of seven levels that are assigned to correspondence thinking. At L4, students can use a single case in order to describe the functional relationship. However, they cannot provide other examples or a general rule of this relationship. For example: “2 x 2 = 4”
- L5 (called *functional-particular*): As extension of L4, students can use multiple concrete examples for describing a functional relationship. Still, they are not able to generalize this relationship. For example: “1 x 2 = 2, 2 x 2 = 4, 3 x 2 = 6, 4 x 2 = 8, …”
- L6 (called *functional-basic*): At L6, students can identify a functional relationship in a general way. However, they still struggle with determining the transformation between the two variables. For example: “Times two”
- L7 and L8 (called *functional-emergent*): Students at these two levels can identify the function rule in a general way but without explicitly connecting the two variables. At L7, they use variables and at L8, they use words for their description. For example: L7 → “d x 2” and L8 → “You multiply the desks by 2.”
- L9 and L10 (called *functional-condensed*): Extending L7 and L8, students at these two levels can describe the functional relationship in a general way and explicitly by connecting the two variables. Again, the lower L9 refers to giving the rule in variables whereas L10 involves a verbal statement on it. For example: L9 → “p = d x 2” and L10 → “If you multiply the number of desks by 2, you get the number of people who can sit.”

Blanton, Brizuela and colleagues (2015) mention a further level as final step, namely the *function as object* level. This level can be characterized as students being sensitive for the limitations of the generality of the identified functional relationship. This means that students recognize that the function rule only works under particular circumstances and cannot be applied if the underlying situation changes.

Empirical data suggest that students show a shift from the lower levels to the higher ones during a particular early algebra intervention: Stephens et al. (2016) report a shift from L0 (*prestructural*) to L8/L9 (*function-emergent* and *function-condensed*) – often involving L2 (*recursive pattern-particular*) and L5 (*functional-particular*) as intermediate levels – within the grades 3 and 4. On a similar empirical base, Stephens et al. (2017, p. 154) identify a *main path* over time from L0 (*prestructural*) via L2 (*recursive pattern-particular*) and L6 (*functional-basic*) to L9 and L10 (*functional-condensed*). They describe this as *main path* because the largest number of students fall into this pattern whereas paths emphasizing recursive and covariation perspectives were identified parallel to this main path but found less often. What should not be disregarded is that students accordingly have less difficulty with expressing a functional relationship by variables than by words.

Besides the research on the development on functional thinking in general, some researchers have focused on the development of covariational reasoning. Although the function-related covariational thinking does not perfectly coincide with covariational reasoning, we can learn from research in this field. Carlson et al. (2002, p. 354) describe covariational reasoning as “the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other.” Thompson and Carlson (2017) propose a framework for covariational reasoning integrating elements of prior research that consists of six levels: *no coordination*, *precoordination of values*, *gross coordination of values*, *coordination of values*, *chunky continuous covariation* and *smooth continuous covariation* (Thompson & Carlson, 2017, p. 435). A description of the six levels is provided in the following:

- *no coordination:* If a student reasons at this level, their focus is only on the variation of one variable at a time and they do not coordinate the values of the variable.
- *precoordination of values:* At this stage, a person is aware that both variables vary but the variation does not occur at the same time. In the person’s mind one variable changes and this first change is followed by a change in the second variable.
- *gross coordination of values:* This level of covariational reasoning describes a stage where a person sees a loose connection between the changing values of two quantities. However, the link is still nonmultiplicative.
- *coordination of values:* At this level, a person is able to coordinate the values of one quantity together with the value of another one, and anticipates to create a discrete connection of pairs.
- *chunky continuous covariation:* The two highest levels of covariational reasoning differ in the kind of variation a person envisions. The person anticipates changes in the quantity of one variable occurring simultaneously with the increase or decrease in the quantity of another variable. At this level, the variation occurs in chunks.
- *smooth continuous covariation:* The difference to the previous level is that the variation is a smooth process.

The first levels of this framework are expected to be mastered by students in Grade 1 to 3 (Pittalis et al., 2020). An emphasis on covariational reasoning can support the development of the correspondence perspective of functions (Ellis et al., 2016), the general development of functional thinking (Stephens et al., 2017) and help build a strong foundation for the understanding of the concept of a function (Thompson & Carlson, 2017). Moreover, strong covariational reasoning skills can also serve as a basis for dealing with functions in upper secondary and tertiary education, e.g. when constructing three-dimensional graphs (Weber & Thompson, 2014).

Figure 2 combines the frameworks by Blanton, Stephens, et al. (2015) resp. Stephens et al. (2017) and Thompson and Carlson (2017). The indicated levels on the top (blue) represent the framework for covariational reasoning (Thompson & Carlson, 2017). The levels indicated on the bottom show possible developmental stages of functional thinking as a whole (Blanton, Stephens, et al., 2015; Stephens et al., 2017). These levels can be combined in four stages: *no functional thinking, variational thinking, covariational thinking and correspondence thinking*. They are followed by thinking in terms of a mathematical object (not indicated in Figure 2). The grey and black icons indicate learning paths as described above. Besides the two frameworks, Figure 2 indicates levels of sophistication within each function aspect over a prototypical time line. The input-output assignment develops first, followed by the aspects of covariation, correspondence and mathematical object. The development of the covariational and the correspondence view lay somewhere between the very concrete and intuitive dealing with functions (input-output) and higher-order activities (mathematical object) (e.g., Blanton, Brizuela, et al., 2015; Pittalis et al., 2020; Stephens et al., 2012; Stephens et al., 2017).

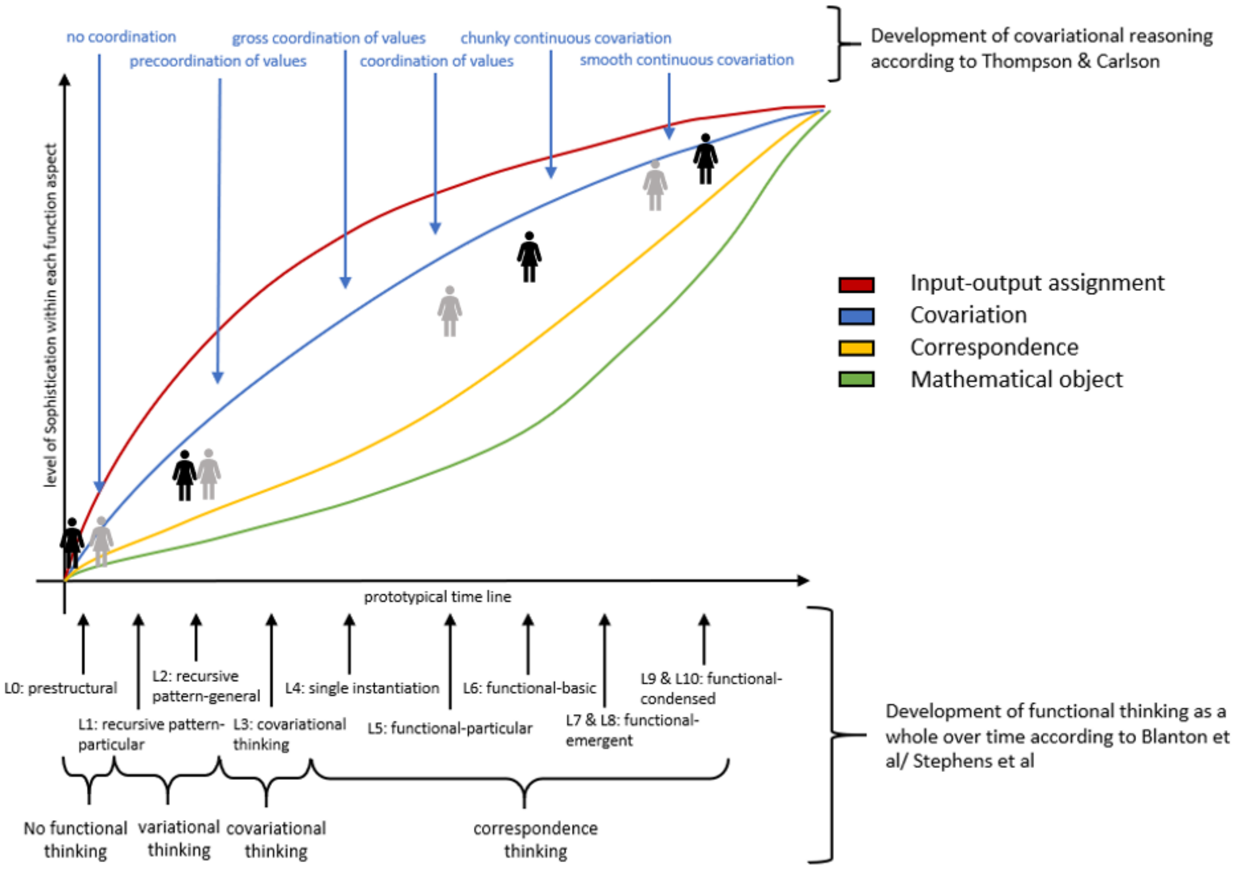


Figure 2: Development of functional thinking

Coming back to the more general model of functional thinking from Stephens et al. (2017), resp. Blanton, Brizuela, et al. (2015), other researchers propose frameworks providing (partly) similar elements. For instance, Pittalis et al. (2020) describe and empirically validate four factors within recursive patterning, namely a) extending repeating geometric patterns, b) finding / expressing the rule of repeating geometric patterns, c) extending growing geometric patterns, and d) extending number patterns. Similar to the framework described by Blanton, Brizuela, et al. (2015) and Stephens et al. (2017), Pittalis et al. (2020) also found covariational thinking and correspondence thinking (in a particular and general way) as subsequent levels. Moreover, their research confirms the existence of two parallel paths: The main path was observed from recursive patterning via correspondence-particular to correspondence-general, whereas the parallel path went from recursive patterning via covariational thinking to a correspondence-general view. Furthermore, the authors claim that a deep understanding of recursive patterns can facilitate correspondence-general thinking. Moreover, Wilkie (2014; see also Markworth, 2010) presents a framework that similarly starts at the lowest level with growing patterns and leads via covariation / recursive generalization and later via correspondence / explicit generalization to an understanding of functional relationships in an object view.

The described typical learning path from no evidence of functional thinking through recursive patterning and covariational thinking to a correspondence view (see Stephens et al., 2017) can not only be observed at the beginning of schooling (primary and lower secondary school) but it can also be transferred to learning processes at higher levels. Therefore, similar courses of learning – of course with regard to a higher and more abstract level of sophistication or more complex content – can be found in higher grades and at university level, for example, when dealing with a range of function classes. In the following, studies which show similar learning paths are highlighted. Ellis et al. (2013; 2016) investigated middle grade student`s (13-14 years old) understandings of exponential growth. The development of the students' understanding appeared in three major stages: prefunctional reasoning, covariational (growth) view, and correspondence (static) view. In most cases, prefunctional reasoning was the first stage, preceding an early understanding of covariation, followed by a more sophisticated view of covariation parallel to a correspondence view of exponential growth. Yet, the correspondence view was influenced by the covariational thinking.

The above-mentioned findings imply a progression of levels of sophistication with regard to functional thinking over time which is linked to appropriate instruction. For instance, the research from Breidenbach et al. (1992) among second year college teacher students has revealed that high levels of sophistication with regard to functional thinking cannot be expected if no sufficient instruction has taken place – not even from university students: more than half of the participating students did not show any understanding of the concept of a function and the understanding of the remaining students was also rather low and far from an object view according to Dubinsky and Harel (1992) before further intervention. Other studies support these results (Dubinsky & Wilson, 2013). This implies – once again – that developing high levels of sophistication with regard to functional thinking does not "happen automatically" but depends on the instruction and learning possibilities at hand. Hence, teachers support is crucial for developing functional thinking.

We will conclude this chapter, which focused on empirical and theoretical insights on the development of functional thinking, with a look on different school curricula in connection to the prominence of function aspects during school time. As mentioned at the beginning of this chapter, the national curricula frame learning opportunities and hence are also an important factor for the development of functional thinking. Figure 3 indicates that the prominence of each aspect varies during the course of school time and provides examples of corresponding topics from FunThink partner countries' curricula. In pre-school, kindergarten, and at the beginning of primary school, mostly the input-output assignment is focused. Later on, in the course of schooling, the focus shifts to a covariational and correspondence view followed by a rather abstract view of function as a mathematical object. The previously focused aspects do not completely disappear but are still needed, just the main focus shifts. For example, during upper secondary education when analyzing functions, the aspect of mathematical object might be mainly focused (e.g., concatenating or differentiating functions) but the aspects of covariation and correspondence are also needed for creating or dealing with corresponding graphs and tables (e.g., determining monotony or the convergence of functions).

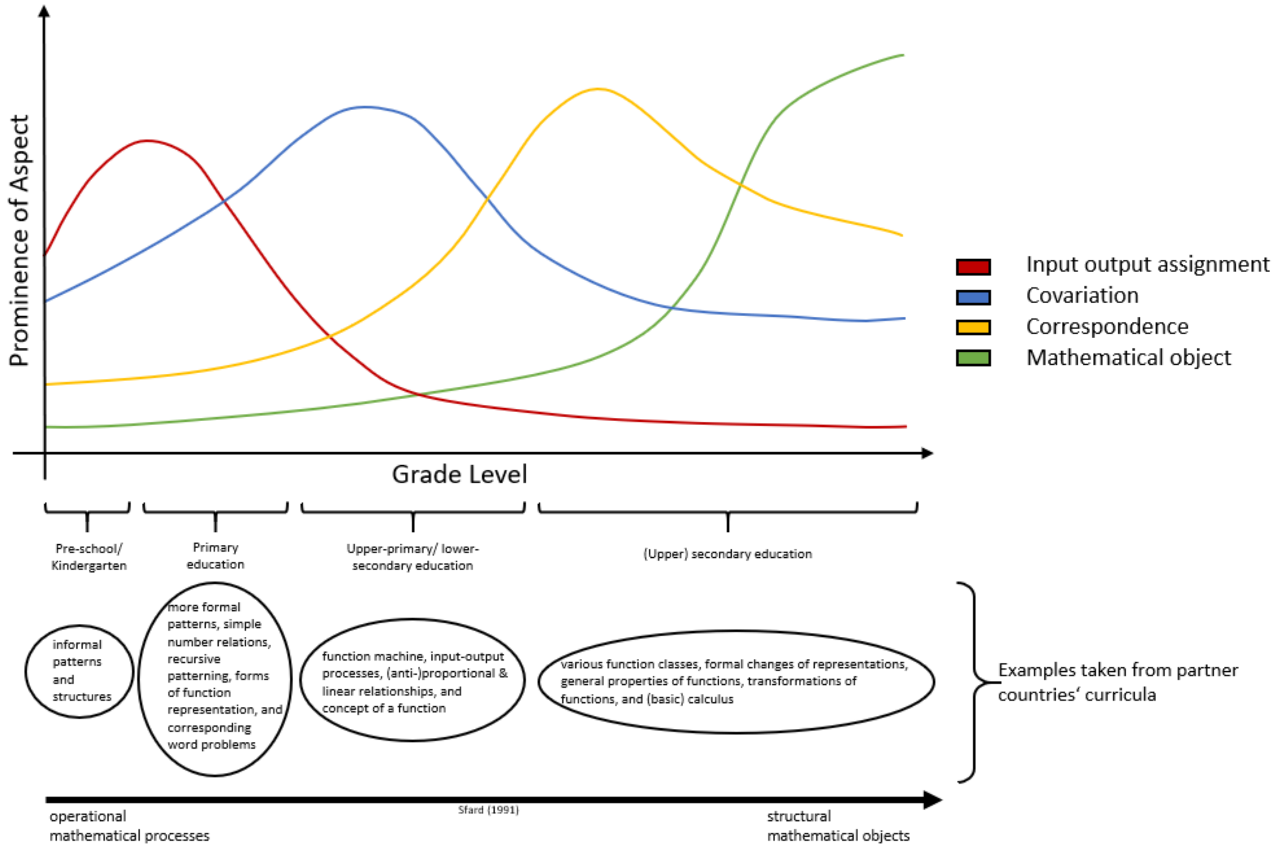


Figure 3: Prominence of the function aspects during school time

As these descriptions show, the curricula of pre-school, kindergarten, and primary education emphasize operational tasks with a focus on the mathematical process. This is essential for the formation of a strong basis for the further study of more abstract definitions of functions which are focused in secondary and tertiary education (see e.g., Pittalis et al., 2020). With the increase of grade levels, the focus shifts to a rather structural view with a focus on mathematical objects (Sfard, 1991). A table with the detailed distribution of treated topics connected to functional thinking for each of the partner countries can be found in chapter 5.

3. Design principles for functional thinking learning environments

3.1 Inquiry-based teaching and learning

3.1.1 Global description

Inquiry is the process of asking a question and trying to answer it. The idea to give such a process a central role in education can be traced back to Dewey (1938) who coined the term *reflective inquiry*. Inquiry-based education is learning and teaching sprouting from a problem situation. Such a situation invites students to engage in activities that are often associated to research. This concerns activities such as formulating questions, study what is known about the situation, form hypotheses, do experiments, observe, reflect, formulate conclusions and new questions (Dorier & Maass, 2020). Research in mathematics has some of its own additional, important mental activities related to the deductive nature of mathematics and its methodology – like defining, posing hypotheses, proving and disproving, formulating theorems and their logical transformation and negation – that could be part of inquiry-based mathematics teaching. Inquiry based education forms an addition to more traditional ways of teaching based on demonstration and repetition.

There is a number of reasons to aim for inquiry-based education. Firstly, inquiry-based teaching stimulates students to be inquisitive humans. Beyond subject-specific learning goals – like being able to differentiate polynomials – general learning goals shape education. Being able to perform any of the previously mentioned inquiry activities could be a learning goal. Secondly, inquiry-based teaching entails knowledge construction. Constructive approaches to teaching address students' sense-making, by allowing students to build new knowledge on situations and knowledge meaningful to them. Meaningful knowledge is more likely to be applied in new situations than less meaningful knowledge, for example learned by memorization. Thirdly, inquiry-based teaching is implemented through collaborative learning. Learning to collaborate in inquiry situations is yet another general learning goal.

Artigue and Blomhøj (2013) discuss various mathematics education frameworks that are suited to address inquiry-based mathematics education: among others, problem solving (Schoenfeld, 1992), *Realistic Mathematics Education* (Freudenthal, 1991), and the *Theory of Didactical Situations* (Brousseau, 1997). *Realistic Mathematics Education* aligns with inquiry-based mathematics education as the process of mathematizing is prioritized over mathematical content. Mathematizing is the process of organizing and re-organizing phenomena by mathematical means (see also Appendix B; Treffers, 1987; 1988). In this sense, mathematizing can be seen as what mathematics researchers – and to some extent science researchers – do. The main distinction is that researchers invent new mathematics, where students usually just reinvent mathematics. Freudenthal describes his pedagogy as *guided reinvention* (1991) and emphasizes the importance of mathematizing situations that are meaningful (see also the section on Situatedness). There are various views on the process of guiding a student during inquiry. Freudenthal mentions it is about "striking a subtle balance between the freedom of inventing and the force of guiding" (p. XXX, 1991). In the problem-solving tradition the guidance is taken as an opportunity to teach students to approach problems through general principles called heuristics (Schoenfeld, 1992). Students are not supported by simply providing a next concrete step, when they are stuck, but instead are invited to use a more general principle or technique. The *Theory of Didactical Situations* suggests instead that, once the problem situation has been presented to the student, the teacher should withdraw. Any problem should be such that it allows students to make progress independently, otherwise the problem situation has failed – not the student. Only after the students have presented their solutions and have discussed amongst themselves whether these solutions are valid, the teacher picks up an active role and connects student solutions to the intended learning goals. Guidance is an important aspect to take into account when designing an inquiry task – various approaches are possible.

3.1.2 Applied to functional thinking

Developing functional thinking does not seem to be a learning goal which is suited for a pedagogy of strictly demonstration and repetition. Grasping the various meanings of functions (see a to d in 1. Functional thinking) requires students to engage in inquisitive ways, and more importantly, functional thinking offers the opportunity for students to do so. Functions in all its aspects, as input-output relations, dynamic processes of covariation, as correspondence relation, and as an object, could be inquired using digital tools in embodied (see: 3.2) learning environments. As the level of abstraction increases and functions are treated as objects it is important to shift to an inquiry of possibilities to (inter)act with functions and properties of functions (or any premature notion of function student may have).

For example, a classical inquiry task on pattern to formula-transitions is depicted in Figure 4. Note how the task is formulated in an open way, not insisting on an approach. Students decide themselves about for example how many more patterns they need or whether they want to make a table. Different students might arrive at solutions in different ways. For example, the left top can be seen as three dots in a row plus four legs sticking out and growing: 3 + 4n. It can also be seen as a rectangle of 3 by 2n+1 minus columns of length n: 3(2n+1)-2n. By letting students compare their solutions one finds a starting point for discussing algebraic manipulation, including the rule of distributivity, and as such can contribute to the development of functional thinking.

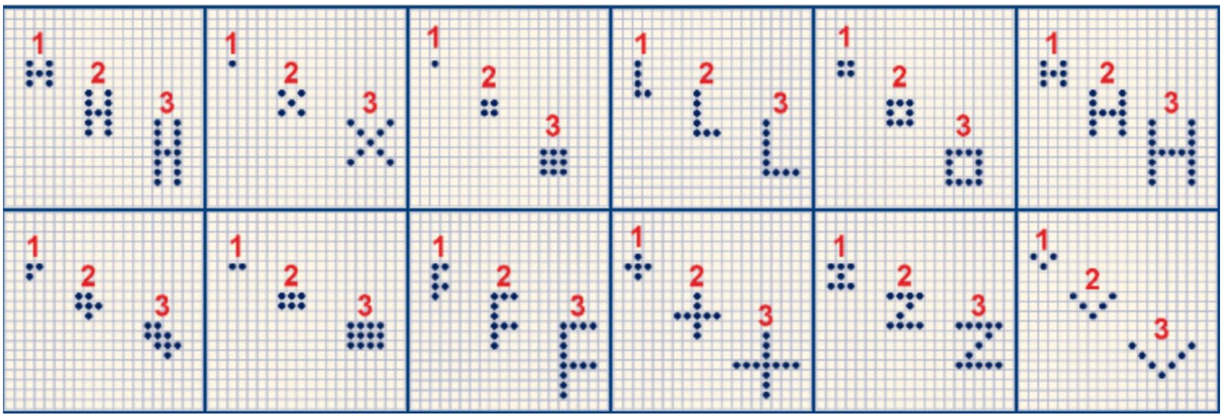


Figure 4: What is the next shape? How many dots does it have? What is a formula for the number of dots in shape number n? (source: www.henkreuling.nl)

3.1.3 Concrete guideline / design principle

Figure 5 shows an overview of essential ingredients of inquiry-based education organized in five themes, taken from the PRIMAS-project (https://primas-project.eu/). In another project, MERIA (https://meria-project.eu/), inquiry-based lesson plans were designed according to a template (see Appendix A) to ensure lessons would include ample time for presenting and discussing students' productions, and the teacher connecting those productions to the learning goal.

In this approach, teachers:

- pose open questions: allow students to engage, explore, explain, extend and evaluate;
- foster an inquisitive classroom culture: value contributions and mistakes;
- build on students' contributions and experiences.

Specifically applied to functional thinking this means that they, for example,

- invite students to investigate the relation between quantities;
- invite students to explore by acting on functions (composing, inverting, graphing, differentiating), shifting the attention to properties of functions (domain, range, continuity, limits, smoothness, invertibility).

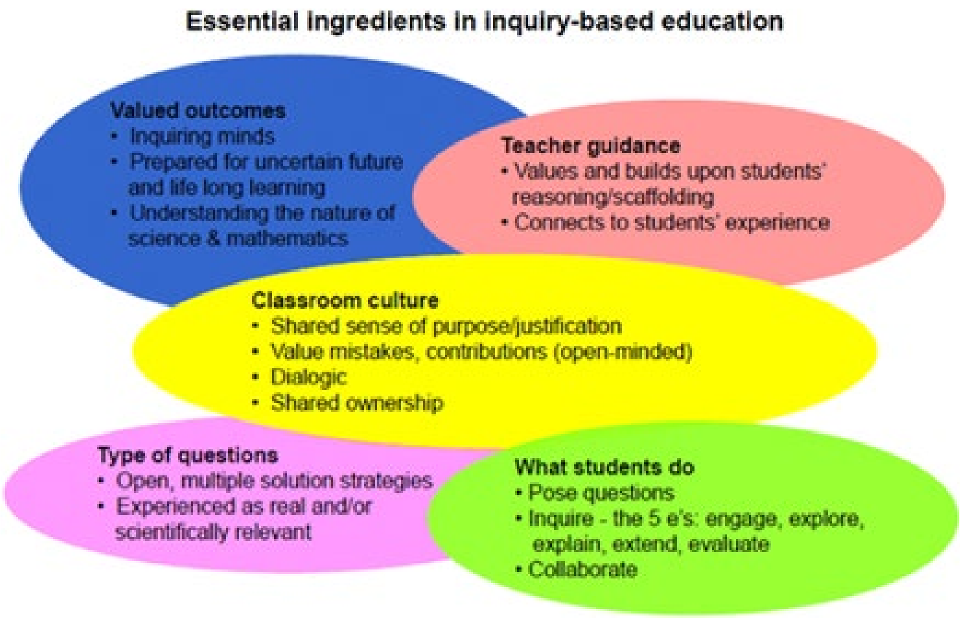


Figure 5: Essential ingredients in inquiry-based education arranged along five themes according to Primas (https://primas-project.eu/)

3.2 Embodiment

3.2.1 Global description

Thinking, or cognition, is traditionally seen as an activity which takes places in the brains of human beings. More modern theories on cognition consider cognition (and thinking) to be embodied, not merely located in the mind of people, but grounded in action-perception experiences with their mind and body (e.g., Lakoff & Nunez, 1999). Theories on embodied cognition, or embodiment, differ in what they consider to be activities with the body, allowing for action-perception experiences or loops (for a taxonomy of such views see Duijzer et al., 2019 or Skulmowski & Rey, 2018). The form of embodied cognition which we follow is that of situated cognition, which means that everything a person experiences, the entire surroundings including, but not limited to, experiences with their own body, seeing others or objects performing actions, imagining such actions, and more are considered part of cognition. Per this view, bodily experiences and interactions with the environment not just support or influence cognition, but these action-perception cycles are essential for and shape it.

In mathematics education this view on cognition has recently found a foothold, with mathematics educators using students' own movements or teachers' movements as part of the learning environment to develop mathematical understanding. One can think of students or teachers jumping on imaginary number lines to develop understanding of number (cf. Menne, 2001) or students acting out shapes of graphs with their arms until more advanced approaches in which digital tools are used to represent movement graphically (e.g. motion sensors, see Duijzer et al., 2019) or tracing unit circles and trigonometric graphs on an iPad (Shvarts et al., 2019). What all these approaches have in common, is that students' learning of a concept is supposed to take place in their bodily interaction with the physical environment. The actions students undertake in these environments to solve problems and learn the concepts are not randomly chosen; they are deliberately related to aspects of the to-be-learned concepts.

3.2.2 Applied to functional thinking

For students to develop functional thinking we will make use of embodied learning environments. For this it is important to make explicit which aspects of functional thinking can be elicited through action-perception experiences, which activities and which actions students undertake are prone to aid their development of understanding of functional thinking. Some of the examples described in the previous paragraph could be useful for this. For example, walking a graph (Duijzer et al., 2019; see Figure) would be focused on developing graphical understanding, understanding of a representation of a functional relationship between distance and time, which students gain by walking in front of a motion sensor and perceiving the resulting graph on a screen. Another example is the use of the unit circle to trace graphs of trigonometric functions on an iPad (Shvarts et al., 2019).

3.2.3 Concrete guideline / design principle

As a concrete guiding principle for the design of student materials that focus on functional thinking, and including an embodied perspective, we will for each of the four aspects of the function concept, and for different age groups, identify

- Tasks and activities, that involve embodied experiences leading to meaningful mathematical cognition concerning that aspect of the function concept;
- Phenomena where mathematical organizing leads to constructing or considering function (in any of its aspects);
- Contextual and cultural aspects of the situation.

3.3 Tool use

3.3.1 Global description

Since the origin of mankind, humans have been using tools to extend their scope and to carry out tasks more easily and more efficiently. Over time, tools have become more sophisticated and have been designed to address cognitive tasks, from different fields, including mathematics (Monaghan et al., 2016). Physical artefacts such as the abacus and compasses facilitated calculations and geometrical constructions, respectively. More recent are electronic tools such as calculators, spreadsheets, statistics software, dynamic geometry software, and computer algebra systems (Drijvers, 2019). Contemporary tools include multitouch technology, motion detectors, augmented and virtual reality environments, and tools that embed artificial intelligence. Now that digital tools are so sophisticated and versatile, they offer means to make students engage in embodied activities to ground their mathematical cognition.

There are different ways to categorize digital technology for doing and learning mathematics. As a first dimension, we distinguish dedicated tools with a limited and specific functionality, such as an applet to build function chains, and general-purpose tools, that offer a wide range of applications. As a second dimension, we notice that some tools may be related to different mathematical domains: there is software for statistics (e.g., Tinkerplots), for geometry (dynamic geometry software), and software for algebra (e.g. computer algebra systems). Third, we can distinguish tools that are specifically designed for educational purpose (e.g., the Numworx learning platform, or GeoGebra) from tools that are used mainly outside education, for example in daily life or in professional settings, such as Excel spreadsheet software, SPSS for data analysis, or Wolfram Alpha. While choosing tools for use in mathematics education, it makes sense to keep these dimensions in mind.

In spite of this wide range of tools available, using them for doing and learning mathematics is not as straightforward as it might seem. Tools are not just "neutral" mathematical assistants that help us to carry out tasks, and as such simplify our lives. Rather, they come with affordances and constraints, and transform mathematical activity (Hoyles, 2018). Therefore, using tools in mathematics education requires paying attention to the subtle interplay between tool use and mathematical learning. A theoretical approach that acknowledges this, is instrumentation theory (Artigue, 2002; Trouche, 2004). In a nutshell, this theory stresses the need for a process of instrumental genesis, that a student goes through while using a tool for doing and learning mathematics. This instrumental genesis comes down to the co-emergence of techniques for using the specific tool for the given task, and the development of mathematical meaning involved in the topic. This approach is key to a fruitful integration of digital technology in mathematics education.

3.3.2 Applied to functional thinking

To apply this instrumentation view to functional thinking, an important step is to identify the different faces of the mathematical meaning involved, as well as the tools, task and actions that might lead to the desired co-emergence of techniques and insights. Taking the above four aspects of the function concept as a starting point—input-output assignment, dynamic process of covariation, correspondence relation, and mathematical object—the question is which digital tools, tasks and actions should be set up to foster the targeted process of meaning making.

To prelude on this design process, Figure 6 shows on the left the screen of an applet which allows for a free creation and use of input-output chains. This clearly matches the *input-output assignment* view described in Section 1. Eventually, such chains can be linked to other representations, such as tables, graphs, and symbolic forms. This may lead to involving other views on function, such as a covariational view, and, if more than one chain is considered, an object view. The right-hand screen in Figure 6 shows a more advanced task, in which students are invited to sketch the graph of the derivative of the function the graph of which if provided. Once the student has finished sketching, the tool provides the primitive function that corresponds to the student's sketch, and this graph can be compared to the original one. This feedback may lead the student to further improve the sketch. In this task, a function, represented by its graph, is considered a mathematical object, which has a corresponding object, namely the graph of the derivative. In the meantime, the activity prepares for the notion of area function and integral.

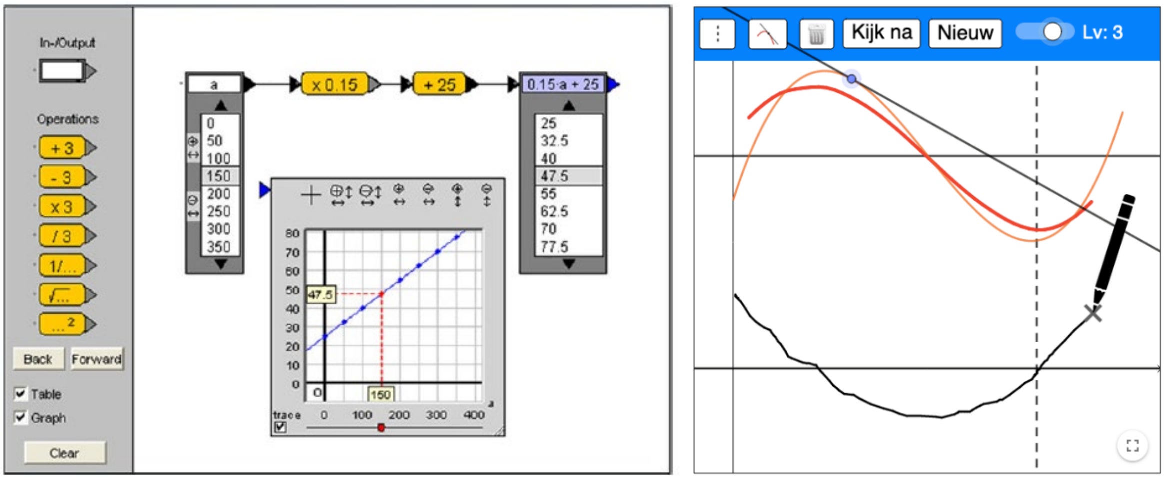


Figure 6: Educational tools to foster functional thinking: on the left the Algebra Arrows Applet, taking the input-output chain as starting point (https://app.dwo.nl/vo/), and on the right an applet to globally sketch the graph of a derivative (https://www.geogebra.org/m/qzzuzsrj#material/axbpeatt).

3.3.3 Concrete guideline / design principle

As a concrete guiding principle for the design of student materials that focus on functional thinking, and including an embodied perspective on using digital tools, we will for each of the four aspects of the function concept, and for different age groups, identify

- Appropriate tools, that offer opportunities for meaning making related to that aspect of the function concept for the targeted student population in a mathematically sound way;
- Appropriate tasks and activities, that involve embodied experiences leading to meaningful mathematical cognition concerning that aspect of the function concept;
- Sets of techniques that emerge in the activity with the tool, that match targeted cognition aspects.

The two screens in Figure 6 show just a first, tentative step in bringing these guidelines into design practice.

3.4 Situatedness

3.4.1 Global description

Freudenthal (1983) proposed the notion of didactical phenomenology. Taking a didactical phenomenological perspective starts by the use of phenomenologically rich situations, that beg to be organized. In such a didactical phenomenology, situations should be selected in such a way that they can be organized by the mathematical objects which the students are supposed to construct (Gravemeijer & Terwel, 2000). This organizing by constructing mathematical objects is in fact mathematizing and includes modelling, symbolizing, abstraction, schematizing, and structuring (Freudenthal, 1973). A consequence of taking a phenomenological viewpoint is that 'knowledge needs to be considered as being situated, meaning that is in part a product of the activity, context, and culture in which it is developed and used' (Brown, Collins, & Duguid, 1989, p. 32). As the learner's body is a part of the situation being mathematized, embodiment needs to be included in situatedness as design principle (Núñez, Edwards, & Matos, 1999). However, there are obvious situations that are not embodied, but situatedness support mathematics learning. This is for example so when a graph describes a 'distant' phenomenon like the distance from the moon to the (center of the) earth over time. Students here are asked to explain the form of the graph and its extremes.

Situatedness refers to meaningful situations in need to be organized or mathematized. Including meaningful in the description makes situatedness depend on what the learner experiences as being meaningful. A formula or abstract functional relation can be meaningful for advanced learners of mathematics. If it is not, we will not speak of situatedness.

3.4.2 Applied to functional thinking

We distinguished four aspects of function: input-output, dynamic process of covariation, correspondence relation and mathematical object. A situated perspective on these aspects includes:

- embed the aspects of function in a phenomenological rich situation, e. g. a process that is in need for organizing, by mathematizing,
- consider the aspects of function as meaningful in their contexts and cultures, where the process the function describes is embodied.

Examples of situatedness in FT

Situatedness gives reason to mathematizing. In FT this mathematizing results in a functional relationship, for example in a graph, or is used for interpreting the functional relationship. In the following example shows this process of mathematizing. Students (aged 10-12 y) are discussing the weather forecast in class. The teacher tells: 'The next three days will be cold, about 9 degrees, but after that temperature in the afternoon will rise till maximum 15 degrees Celsius. The nights remain somewhat cool, with only 5 degrees.' The students are asked to sketch a graph depicting the situation. The graphs are shared and discussed.

In another activity students are given a graph. The x-axis of this graph is a 24-hour timeline. The graph shows how happy a specific child is during the day. In group work students discuss what could have happened over the day.

Embodiment is not explicit in these examples; however, a closer look shows aspects of embodiment. In the first example embodiment comes forward when students sketch a graph, where an upward pencil movement corresponds to the temperature going up and a downward movement represents cooling down. In the second example group work elicits communication on the graph, which next could lead to pointing to the graph and its development in time explaining arguments on what the graph could be about.

In section 3.2.2 the Duijzer et al (2019) experiments are described, which combine situatedness, embodiment and tool use, by walking a graph.

3.4.3 Concrete guideline / design principle

Make phenomenologically rich situations accessible for functional thinking, by:

- selecting phenomena where mathematical organizing leads to constructing or considering function (in any of its aspects),
- considering context and cultural aspects of the situation,
- offering means for mathematizing,
- including embodiment.

4. Literature review on functional thinking

The FunThink vision document was grounded in a scoping literature study. In this section, we provide a comprehensive overview of the resulting literature on functional thinking, embodiment, digital technology, and abstraction/reification in mathematics education research. We synthesize how research literature informs our work on embodied approaches to functional thinking using digital technology as to foster abstraction, and, as such forms the foundation of the FunThink vision on functional thinking education. Appendix C describes our approach to set up this literature synthesis.

4.1 Functional thinking

Functional thinking has been discussed as a critical area of mathematics throughout primary, secondary, and tertiary education since the beginning of the twentieth century (Vollrath, 1986). Although there is no widely adopted definition of functional thinking, we propose that functional thinking encompasses the process of building, describing, and reasoning with and about functions (Pittalis et al., 2020; Stephens et al., 2017). In a broader interpretation, functional thinking connects to the four main aspects of function distinguished in literature (that also forms the backbone of Section 1):

a. *Function as an input-output assignment:*
   The function is an input–output assignment that helps to organize and to carry out a calculation process, in which pattern recognition related to pre-algebraic thinking is regarded as a first step.
b. *Function as a dynamic process of covariation*
   The independent variable, while running through the domain set, causes the dependent variable to run through a range set. The dependent variable co-varies with the independent.
c. *Function as a correspondence relation*
   This view on function concerns understanding the relation between the independent and the dependent variable and being able to represent it. This includes the mapping view and may lead to the more formal definition of function as a set of ordered pairs.
d. *Function as a mathematical object*
   A function is a mathematical object which can be represented in different ways, such as arrow chains, tables, graphs, formulas, and phrases, each providing a different view on the same object.

In this section, we have seventy papers on functional thinking, of which some also address embodiment, digital technology, and abstraction/reification. The most oft-mentioned aspects of functions are the dynamic process of covariation (e.g., Carlson et al., 2002; Johnson et al., 2017; Thompson & Carlson, 2017) and correspondence relations (e.g., Blanton & Kaput, 2005; Dreyfus & Eisenberg, 1982; Smith, 2003). This reflects the main focus of researchers on studying the relation between two varying quantities/variables, such as, "A function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values…" (Thompson & Carlson, 2017, p.444) and "In a correspondence approach, the emphasize is on the relation between corresponding pairs of variables. … In the covariational approach, the focus is on corresponding changes in the individual variables" (Erick Smith, 2017, p.146).

Table 1 and Figure 7 provide an overview of the seventy-one studies on functional thinking included in the review in terms of education levels and the function aspect being addressed. In this section, there are forty-five articles from ONLY functional thinking category, nine articles from Embodiment category, ten articles from Digital Technology category and seven articles from Abstraction category. About half of the studies (n =39) involve more than one aspect of function when discussing functional thinking. Besides, over all the three education levels, more studies are aimed at primary education and secondary education. We observed a much higher frequency of studies that involved aspect of input-output assignment at the primary education level (21 out of 31), which may relate to the corresponding curricula system. Moreover, algebraic thinking is also a focus of primary education. Functional thinking is regarded as a "component of algebraic thinking based on construction, description and reasoning with and about functions and their constituents" (Pinto & Cañadas, 2018, p. 4-49) that ranges from specific relationships to generalizing the relationships between two (or more) variables (Smith, 2017).

Table 1 Function aspects involved in studies of the literature corpus

Aspects	Primary	Secondary	Tertiary	*n*
Function as an input-output assignment	Payne, 2012; Kimberly, 2012; Stephens et al., 2012; Stephens et al., 2017; Martinez & Brizuela, 2006; Warren et al., 2007;Wilkie, 2014; Tanişli, 2011; Wilkie, 2016; Mouhayar & Jurdak, 2015; Ross, 2011;Beatty et al., 2013; Muir et al., 2015; Warren & Cooper, 2006; Ellis, 2007; Switzer, 2013; Cañadas et al., 2016; Pittalis et al., 2020; Wilkie & Clarke, 2016; Afonso & Auliffe, 2019; Stephens et al., 2016	Ferrara & Ferrari, 2020; White, 2009; Doorman et al., 2012; Jon, 2013; MacGregor & Stacey, 1995; Wilkie, 2016; Wilkie, 2020	Kamber & Takaci, 2018; Asghari et al., 2011; Paz & Leron, 2009	31
Function as a dynamic process of covariation	Blanton et al., 2015; Tanişli, 2011; Payne, 2012; Kimberly, 2012; Nicole & Alan, 2016; Stephens et al., 2012; Stephens et al., 2017; Warren et al., 2007; Panorkou & Germia, 2020; Arzarello et al., 2005; Pittalis et al., 2020	Lichti & Roth, 2019; Fonger et al., 2016; Johnson, 2012; Ellis et al., 2018; Ellis et al., 2013; Wilkie, 2014; Ayalon & Wilkie, 2020; Moore, 2014; Erik, 2014;Vollrath, 1986; Abrahamson & Bakker, 2016;Swidan et al., 2020; Hoffkamp, 2011; Lindenbauer, 2019; Günster & Weigand, 2020; Doorman et al., 2012	Nagle et al., 2013; Paoletti, 2020; Carlson et al., 2002; Hatisaru & Erbas, 2017; Habre, 2017; Nemirovsky & Noble, 1997	33
Function as a correspond	Blanton et al., 2015; Tanişli, 2011; Wilkie, 2016; Mouhayar &	MacGregor & Stacey, 1995; Lichti & Roth, 2019; Fonger et al., 2016;	Nagle et al., 2013; Paoletti, 2020; Asghari	38

ence relation	Jurdak, 2015; Payne, 2012; Ross, 2011; Warren et al., 2006; Kimberly, 2012; Nicole & Alan, 2016; Caddle & Brizuela, 2011; Stephens et al., 2012; Stephens et al., 2017; Arzarello et al., 2005; Heuvel-Panhuizen et al., 2013; Pittalis et al., 2020; Stephens et al., 2016; Pinto & Cañadas, 2018	DeJarnette et al., 2016; Ayalon & Wilkie, 2020; Abdullah, 2010; Erik, 2014; Vollrath, 1986; Abrahamson & Bakker, 2016; Lagrange & Psycharis, 2014; Swidan et al., 2020; Hoffkamp, 2011; Lindenbauer, 2019; Günster & Weigand, 2020; Doorman et al., 2012; Wilkie, 2016a; Ronda, 2015	et al., 2011; Nemirovsky & Noble, 1997	
Function as a mathematical object	Blanton et al., 2015; Tanişli, 2011; Mouhayar & Jurdak, 2015; Ross, 2011; Kimberly, 2012; N Nicole & Alan, 2016; Caddle & Brizuela, 2011; Stephens et al., 2012; Martinez & Brizuela, 2006; Switzer, 2013; Pinto & Cañadas, 2018	Lichti & Roth, 2019; Bulková & Čeretková, 2019; Fonger et al., 2016; Acevedo et al., 2014; Kop et al., 2017; Erik, 2014; Nemirovsky et al., 2013; Günster & Weigand, 2020; Doorman et al., 2012; Ogbonnaya, 2010; Ronda, 2015	Kamber & Takaci, 2018; Nagle et al., 2013; Paoletti, 2020; Hatisaru & Erbas, 2017; Farmaki & Paschos, 2007; Botzer & Yerushalmy, 2008; Julie & Kathy, 2011; Asghari et al., 2011;	30

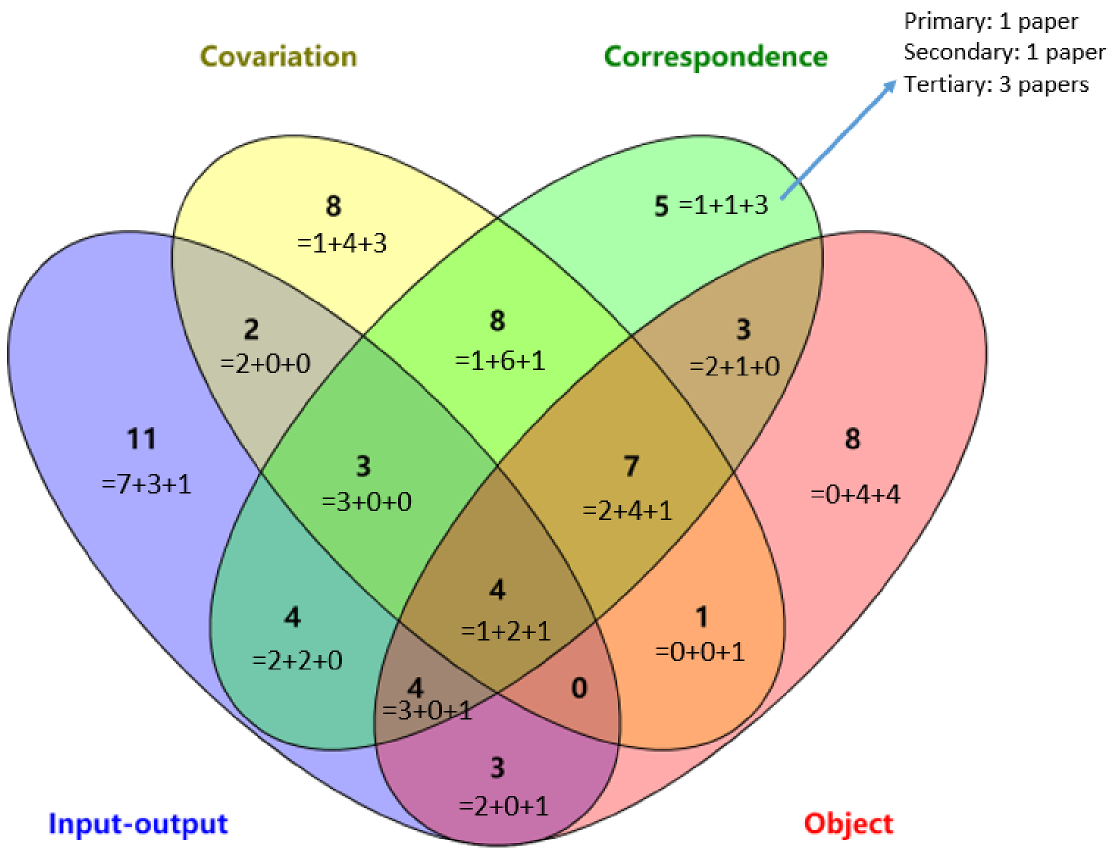


Figure 7: The Venn diagram of four aspects of function over education levels

Levels of functional thinking

The main theoretical discussions of functional thinking concentrate on the level and structure of functional thinking. Stephens et al. (2017) followed Clements and Sarama (2014) in defining levels of sophistication in thinking and described the shifting levels of sophistication observed in student's generalizations and representations of functional relationships through their written responses to the given assessment items. While some researchers dive into specific areas of functional thinking, for example, Carlson et al. (2002) provided (Piagetian) levels of covariational reasoning development-with five levels distinguished: coordination (Level 1), direction (Level 2), quantitative coordination (Level 3), average rate (Level 4), instantaneous rate (Level 5). Furthermore, an individual's covariational reasoning ability reaches a given level of development when it supports the mental actions associated with that level and the actions associated with all lower levels. Interestingly, we found some studies question the commonly described aspects of functional thinking and propose that functional thinking should be seen as a one-dimensional construct. For example, Lichti and Roth (2019) created a test to verify that the construct of functional thinking psychometrically one-dimensional. They partially attributed their striking result to students' age and gender. It shows us a new angle of view in researching the construct of functional thinking with regard to gender differences and age differences. This test might be used in the FunThink project to assess learning gains in the teaching experiments.

Task design for functional thinking

Considering the practical purpose of our project, we also focus on the design and implementation contribution of relevant studies. Several papers have provided well-designed tasks for function learning, such as the growing circles' task: find the relationship between day and the number of circles (Stephens et al., 2017), the classical bottle filling problem: graph the height of the water against the amount of water in the given bottle (Carlson et al., 2002), and the Zig-Zag Functions tasks: a series of open-ended problems with the family of periodic kink functions (Freudenthal Institute, http://www.fisme.science.uu.nl/wisbdag/opdrachten/assignment2010.pdf). And some studies adopted the idea of embodied cognition (e.g., Stylianou et al., 2005; Vollrath, 1986. See details in section 4.2) and used technology-enhanced pedagogy (e.g., Brown, 2015; Doorman et al., 2012; Hoffkamp, 2011. See details in section 4.4).

4.2 Embodiment and functional thinking

From the perspective of embodied cognition, body and mind cannot be separated and a dualistic view on cognition is inappropriate. There are various approaches to interpret embodiment. Based on the Conceptual Metaphor Theory in cognitive linguistics (Lakoff & Johnson, 1980), Lakoff and Núñez (2002) believe cognition emerges through bodily experiences in interaction with the environment to carry out a task or reach a particular goal. Some studies carry a similar idea about embodiment concerning embodied design in function learning (e.g., Font et al., 2010; Oehrtman et al., 2019; Paz & Leron, 2009). From a perceptual perspective, Barsalou frames embodiment through grounding experiences, which is also advocated by Schwartz (1999) and Abrahamson et al. (2016) and employed in their own research. While some researchers take a radical view on embodied cognition. Radical embodied cognitive science differs from other kinds of embodied cognitive science primarily in its rejection of mental representations and mental computations as explanatory tools (Antoniadis & Chemero, 2020). In most varieties of embodied cognitive science, bodily motions and environmental resources are taken as supplementing, or even transforming, the mental representations and computations that are presumed to constitute cognition (e.g., Clark, 1997). In radical embodied cognitive science, mental representations and computations are not taken to constitute cognition. The explanatory work done by mental representation and computation in embodied cognition is replaced by Gibsonian ecological psychology (Gibson, 1979) and nonlinear dynamical systems theory (Port & van Gelder, 1995).

The dataset in this section contains thirteen studies, reporting on embodied designs that develop students' functional thinking or remove barriers in the way of learning function. To achieve the expected development of functional thinking, students need guidance to take action in ways that simulate the core mechanism and spatial relations to enact functional metaphors for the target knowledge domain (Abrahamson & Lindgren, 2014). Abrahamson defined the term embodied design, which was first proposed by van Rompay and Hekkert (2001), as an approach demonstrated helpful in guiding the design tools for student construction of meaning (Abrahamson, 2009). As the embodied designs prepare students for correct intuitive responses or performances before showing them the analytic procedures that validate yet enhance these intuitions (Abrahamson et al., 2020a), we attach importance to embodied designs in mathematics function-specifically, the engagement of activity of embodied design (Table 2) and the type of feedback of embodied design (Table 3).

Categories for embodied task design

Bos et al. (submitted) propose a categorization of tasks based on the engagement of activity of embodied design: action-based, perception-based, and incorporation-based tasks – the latter a new type (Table 2). The former two types, as developed by Abrahamson, reflect researchers' initial investigation that draws on learners' motor coordination and perceptual capacity. Action-based designs aim to ground mathematical concepts in students' natural capacity to adaptively solve sensorimotor problems. Perception-based designs aim to ground mathematical concepts in students' natural perceptual ability, in their naïve views concerning a situation. Similar to the action-based genre, this is followed by a phase of reflection in which these views are developed. Incorporation-based designs are in a sense the opposite of outsourcing a task to an artefact instead of a person. Students are first invited to solve a sensorimotor task with feedback of some artefact (e.g., an action-based task) or observe an artefact's perceptual qualities (e.g., perception-based task), and then invited to perform the same task without the artefact, just with their body (Bos et al., submitted; Shvarts et al., 2021). From Table 2, we can see the action-based designs occupy the majority nowadays.

Table 2 Engagement of activity of embodied design

Action-based (n = 11)	Perception-based (n = 2)	Incorporation-based (n = 1)
Ferrara & Ferrari, 2020; Arzarello et al., 2005; Nemirovsky & Noble, 1997; Botzer & Yerushalmy, 2008; Julie & Kathy, 2011; Nemirovsky & Kelton, 2013; Abrahamson & Bakker, 2016; Paz & Leron, 2009; Leinhardt et al., 1990 (Greeno and Russell & Friel 's research); Stylianou et al., 2005; Ellis & Grinstead, 2008	Vollrath, 1986; Ferrara & Ferrari, 2020	Botzer & Yerushalmy, 2008 (It has aspects of an incorporation task in the sense that on the computer aspects are outsourced)

However, according to some pioneering exploration, the artefact could become incorporated in the constitution of new instrumented actions (Drijvers, 2019). The new type of embodied design, incorporation-based task, characterize outsource a task to a person instead of an artefact (Bos et al., submitted; Shvarts et al., 2021). This new development may help teachers, teacher educators, policymakers and other stakeholders better understand the relationship between learners and artefacts to avoid students becoming 'the slave of artefacts.' Based on this idea, we found that there are several designs regarding physically controlling one or two variables with visual feedback through a graph or a colored frame (e.g., Abrahamson & Bakker, 2016; Botzer & Yerushalmy, 2008; Ferrara & Ferrari, 2020; Nemirovsky et al., 2013; Nemirovsky & Noble, 1997). In these embodied designs, direct feedback goal-oriented actions (Abrahamson et al., 2020a) lead students to "perform" a certain relation between two quantities, which is then mathematized. Besides, some studies construct artefacts that allow students to experiment with the relation between two variables in a physical setting (e.g., Arzarello et al., 2005; Vollrath, 1986), yet there is no direct link to the graph and no direct feedback in these two designs.

Two types of feedback in task design are reviewed in this section: real-time feedback and delayed feedback. The core point is whether there is real-time feedback (with mathematical meaning) on the movement (Table 3). From Table 3, we can see that some tasks with real-time mathematically laden feedback are proven to support students' understanding of function concepts. For example, graphing motion technology, which allows working with couples of positions over time graphs, can provide students with the opportunity to observe the real-time sum of two functions on the screen (Ferrara & Ferrari, 2020). These consistent findings from multiple studies suggest that the type of feedback and the degree of engagement of activity are correlated with levels of kinesthetic and perceptual experience supporting a concrete understanding of abstract concepts (Ferrara & Ferrari, 2020).

Table 3 Feedback of embodied design

Real-time feedback (n = 5)	Delayed feedback (n = 4)
Ferrara & Ferrari, 2020; Nemirovsky & Noble, 1997; Nemirovsky & Kelton, 2013; Abrahamson & Bakker, 2016; Stylianou et al., 2005	Vollrath, 1986; Arzarello et al., 2005; Botzer & Yerushalmy, 2008; Julie & Kathy, 2011

4.3 Digital technology for functional thinking

Cutting-edge technologies with their unique interactions provide a technical carrier for creating the embodied learning environment and open up a wider application space for function learning. By reviewing the oft-used researched forms of technology-enhanced instruction and the main didactical functions for technology in mathematics education, this section can provide the analysis necessary to guide technology integration in an effective and precise manner. We have thirty-five studies in the corpus of digital technology, with sixteen of them having an in-depth discussion of functional thinking and the others only addressing functional thinking to a limited extent.

Since the term "digital technology" has been used loosely in mathematics education, it is important to provide a working definition. In our review, digital technology refers to technology-enhanced instruction that helps to provide learning materials and support learning processes in function learning classrooms. The oft-used types of digital technology in function learning classrooms include computer-assisted instruction (CAI), dynamic geometry software (DGS), computer algebra system (CAS) (Table 4). DGS is very popular among all studies, which provides an environment in which geometric figures and the graphs of function can be easily constructed, manipulated (e.g., dynamic dragging), and measured (Lagrange & Psycharis, 2014) for designing, teaching and learning various functional paradigms.

Table 4 Researched forms of technology-enhanced instruction

Digital technology types	Description	Articles	n
Computer-Assisted Instruction (CAI)	CAI refers to the use of computers to submit or supplement the traditional, teacher-directed instruction.	Thompson et al., 2017; McCulloch et al., 2020; Leinhardt et al., 1990; Ayers et al., 1988; Liang & Moore, 2020; Reed et al., 2010; Getenet & Beswick, 2014; Heuvel-Panhuizen et al., 2013; Arzarello et al., 2005	9
Dynamic Geometry Software (DGS)	DGS refers to a certain type of software predominantly used for the construction and analysis of tasks and problems in geometry.	Godwin & Beswetherick, 2003; Ferrara & Ferrari, 2020; Botzer & Yerushalmy, 2008; Lagrange & Psycharis, 2014; Hoffkamp, 2011; Roux et al., 2015; Caglayan, 2015; Zulnaidi & Zamri, 2017; Günster & Weigand, 2020; Ogbonnaya, 2010; Eu, 2013; Miranda & Sánchez, 2019; Rolfes et al., 2020; Abrahamson & Bakker, 2016	14
Computer Algebra System (CAS)	CAS refers to a certain type of calculator/software that performs calculations, symbolic manipulations and offers graphical representations.	Nemirovsky & Noble, 1997; Hong & Thomas, 2015; White, 2009; Brown, 2015; Kathleen Heid et al., 2013; Doorman et al., 2012; Asli, 2016; Jon, 2013; Nemirovsky & Noble, 1997; Stylianou et al., 2005	8

From the foregoing review, some technologies enhance computational efficiency, and others provide multiple mathematical representations. The major consideration for technology used or design in function classrooms is how to combine the different didactical functionalities for technology in specific mathematical tasks. Drijvers et al. (2011) proposed three main didactical functions of technology in mathematics education: (1) doing mathematics, (2) practicing skills, (3) developing conceptual understanding. Figure 8 shows a schematic overview of the three main didactical functions of technology. Based on the viewpoint of Drijvers et al., the three didactical functions of digital tools are not mutually exclusive but are intertwined. We provide some examples of the didactical functionality of common instructional technologies used in function learning classrooms in the followings.

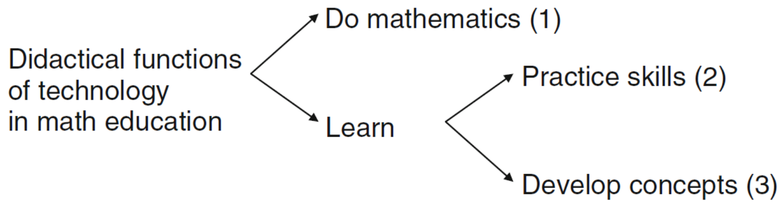


Figure 8: Didactical functions of technology in mathematics education (Drijvers, 2015, p.3)

Doing mathematics

Initially as a tool for outsourcing algebraic procedures while doing algebra, technology can support doing algebra by improving computational efficiency. The functionality of doing algebra also emphasizes the pattern recognition and generalization aspect of function. Different kinds of calculators and software are available, ranging from Excel, scientific calculators, graphing calculators, AlgebraArrows (Doorman et al., 2012), TI-Nspire computer software (Asli Özgün-Koca, 2016) etc. These technology enhancements can help students efficiently complete tasks that could be completed by hand as well as find patterns while dealing with dynamic input-output dependencies.

Practicing skills

The functionality of practicing skills focuses on the technology-enhanced learning environment that can respond to students' solutions and strategies when practicing algebraic skills through intelligent, diagnostic feedback. Some dynamic software provides the possibility to let students receive feedback immediately, for example, the applet *Solving Equations* (Drijvers et al., 2011) and the software *WiiGraph* (Ferrara & Ferrari, 2020). These technology enhancements often "contains didactical choices through the structure and sequence of algebra tasks" (Drijvers et al., 2011, p.183).

Developing conceptual understanding

To evoke specific thinking processes and to guide the development of the students' functional thinking, technology can function to visualize a concept in a dynamic way and generate examples that arouse students' interest and invite generalization or investigation of relationships or properties (Drijvers et al., 2011). Current and emerging software applications and technology, such as Geometer's Sketchpad (GSP), Cabri, Geogebra, Augmented Reality (AR) and Virtual Reality (VR) enable students to manipulate and explore mathematics conceptually. In the domain of function, the DGS software *Cinderella* has been proven to be effective in emphasizing the dynamic view of functional dependencies (Hoffkamp, 2011), and GeoGebra can provide students with the opportunity to engage in an operative process by which students' functional thinking can be developed and enhanced (Günster & Weigand, 2020). In addition, some mathematical exploration and experimentation environment, with the support of technology, also helps visualize abstract mathematical concepts and enables students to understand them in a novel and meaningful way (Nemirovsky & Noble, 1997).

4.4 Abstraction in functional thinking

For the purpose of our project, we describe abstraction as a meaning generating process in which students distil the core patterns within problems and solutions, and thus form the understanding of function. And reification is a constructive process based on generalizing purposefully identified similarities of function (Mitchelmore & White, 2007). In the following review, we analyzed different abstraction stages leading to the formation of function concept and different abstraction levels of students' understanding of function. There are nineteen articles been reviewed in this section.

When discussing the abstraction stages leading to the formation of function concept, most studies lie between the process-object perspective and property-oriented perspective. From the process-object perspective, students first conceive of function as a computational process, and then they start to think of function no longer as a set of procedures but as a mathematical object in itself that they can act on, operate on, or transform (Ronda, 2015; Sfard, 1991; K. J. Wilkie & Ayalon, 2018). Breidenbach et al. (1992) suggested that students' understanding of functions can be considered as moving from an initial focus on actions and processes to more object-oriented views characterized by a gradual focus on structure, incorporation of properties and reification of mathematical objects. Therefore, the covariation aspect of function has been emphasized gradually (Carlson et al., 2002; Confrey & Smith, 1995; Stephens et al., 2017; Thompson, 1994). Another viewpoint highlighted the properties of function. The property-oriented view pointed out that the first stage involves the ability to realize the equivalence of procedures across representations; in the second stage students can translate procedures across representations (Stage1), but are also beginning to realize that some of these procedures have analogues in other function classes; the final stage extends student's ability to identify functional properties, in which students can understand the procedural networks as permanent constructs (Slavit, 1994).

Abstraction has levels. In both the general mathematics education field and the specific function domain, researchers theorized the different layers of abstraction. Hiebert and Lefevre (1986) distinguished the two levels of abstraction to establish the relationship between pieces of mathematical knowledge: primary level and reflective level. The main difference between the two levels is the way relationships are constructed. In the reflective level, relationships are constructed at a higher, more abstract level than the pieces of information they connect in the primary level. In the domain of function, Blanton et al. (2005) proposed eight levels of sophistication in children's thinking about functional relationships: pre-structural, recursive-particular, recursive-general, functional-particular, primitive functional general, emergent functional general, condensed functional-general, function as object. On the basis of their work, Stephens et al. (2016) separated a level, condensed functional-general, into two levels and reordered the levels: recursive pattern-particular, recursive pattern-general, covariational relationship, functional-particular, functional-basic, functional-emergent in variables, functional-emergent in words, functional-condensed in variables, functional-condensed in words. These two ways of the segment of levels both aimed at the primary education level. Further discussions regarding secondary and tertiary education levels are needed in the future.

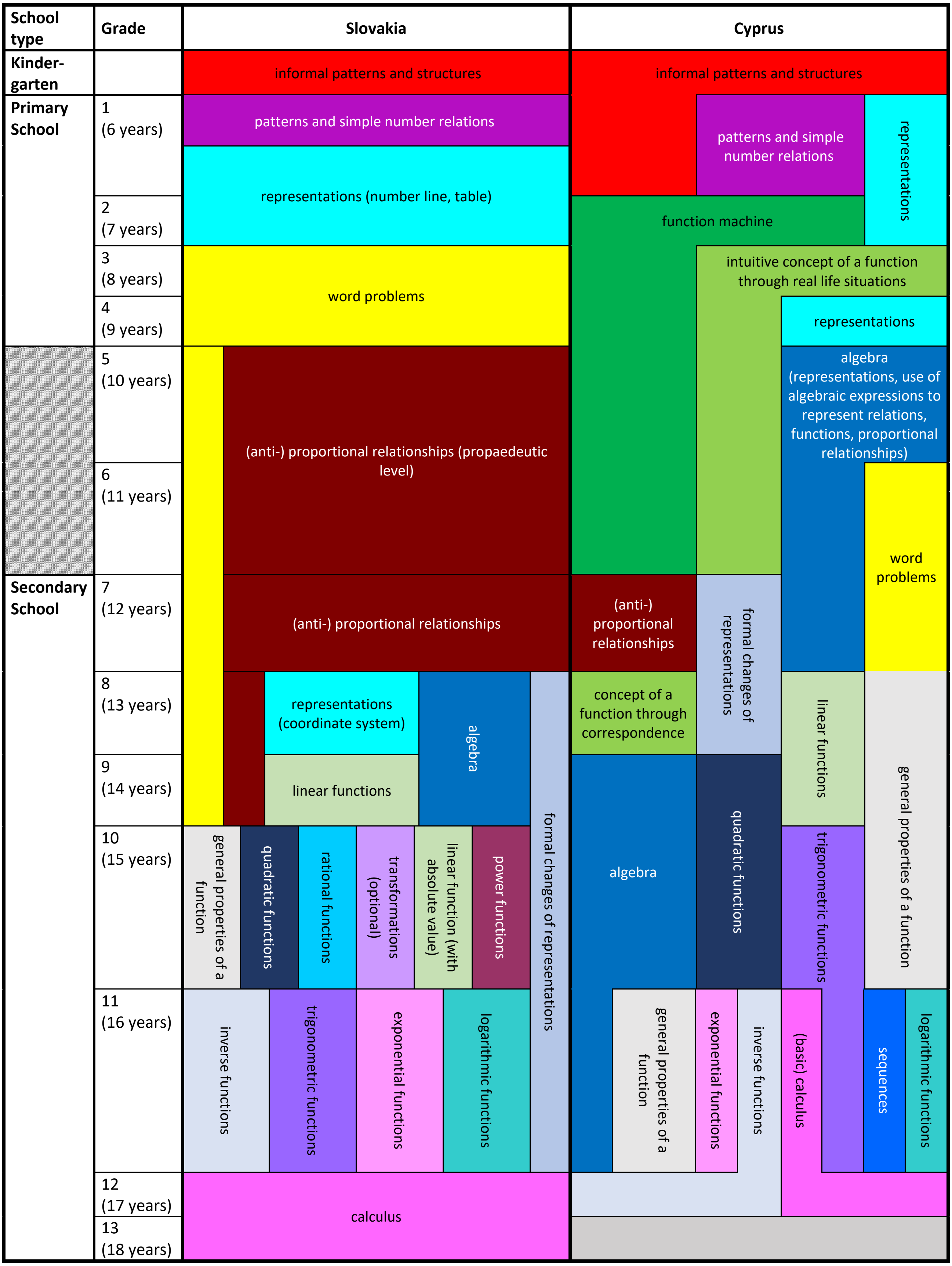
5. Functional thinking in the participating countries' curricula

As indicated in the previous chapters, beyond theoretical considerations and empirical insights, the national curricula play an important role in the development of functional thinking, as they frame what students (ought to) learn about functions. Hence, these curricula have to be considered when learning environments are developed. The following table provides an overview of functional thinking related topics in each partner country's mathematical curriculum. The different school systems of the five countries are presented in a combined table. It is important to note that some countries have schools with different levels of sophistication. The table only includes the highest form of schooling in each country. This decision was made to obtain the most comprehensive overview of all relevant topics. Therefore, the table does not claim for exact fitting to every type of school. Moreover, only topics that have a clear relation to functional thinking are listed and topics with a more indirect relation are left out; this may vary among the different countries. Due to local differences in the curricula within a country, discrepancies are possible. The size of a topic in the table does not reflect its relevance or the time scheduled for it.

The indicated age levels may vary. Kindergarten starts in Poland, Slovakia and Germany at the age of three, whereas in the Netherlands children attend Kindergarten when they are four and five years old. Attending Kindergarten is not compulsory in all countries. Primary school starts at the age of six in Germany, the Netherlands, and Slovakia. In Poland, parents decide if their children start with primary school at the age of six or seven and primary school ends after grade eight before secondary school starts in grade nine. In Cyprus, children are usually six and a half years old at the beginning of first grade.

In Germany and in the Netherlands, topics are treated in a telescopic way. This means that a topic is introduced at a certain point of time (indicated in the table) and then reappears at higher grade levels and is introduced and practiced in more depth and detail (not indicated in the table). Grey shaded squares indicate grade levels in which no new topic is introduced, but previously dealt topics can reappear in another context and in more detail.

School-type	Grade	Germany	Netherlands	Poland
Kinder-garten		informal patterns and structures	informal patterns and structures; picture graphs	informal patterns and structures
Primary School	1 (6 years)	no clear attribution of competences to the grades; patterns and simple number relations; representations	patterns and simple number relations; picture graphs; introduction to representations (empty number line, 'rekenrek', histograms, …)	grades 1-3: no clear attribution; patterns and simple number relations; initial introduction to representations (arrows, arrow graphs, empty space instead of x)
	2 (7 years)	word problems	word problems	word problems
	3 (8 years)	function machine	introduction (continued) to representations (line graphs)	drawing symmetrical figures
	4 (9 years)	function machine (continued)	introduction (continued) to representations (line graphs) (continued)	grades 4-6: no clear attribution; introduction to representations (continuation: number line, histograms, tables)
	5 (10 years)	function machine	introduction function machine	descriptive statistics (introduction, graphs, diagrams)
	6 (11 years)	representations (coordinate system)	introduction (continued) to representations (sector graphs (percent/ fraction), ratio table)	word problems; algebra (representations, algebraic expressions, linear equations)
Secondary School	7 (12 years)	concept of a function; (anti-) proportional relationships; representations (table, graph, text)	proportional relationships; linear functions; function machine; formal changes of representations	grades 7 & 8; proportional relationships (only numerical, without graphs); symmetries; coordinate system
	8 (13 years)	linear functions; formal changes of representation	quadratic functions; power functions; (square) root functions	(continued)
	9 (14 years)	quadratic functions	exponential functions; polynomial functions; general properties of a function	concept of a function; general properties of a function; linear functions; formal changes of representations; absolute value functions; root functions; basic transformations (without vectors)
	10 (15 years)	trigonometric functions; power functions; root functions; exponential functions; logarithmic functions	trigonometric functions; inverse functions; (basic) calculus; absolute value function	grades 10 & 11; quadratic functions; trigonometric functions; power functions; exponential functions; sequences; polynomial functions
	11 (16 years)	polynomial function; (basic) calculus; transformation of functions; statistics; probability	logarithmic functions; transformations	(basic) calculus; logarithmic functions; rational functions; antiproportional relationships; transformations; inverse functions
	12 (17 years)			calculus; probability; statistics
	13 (18 years)			


School type
Grade
Slovakia
Cyprus
Kindergarten
Primary School
Secondary School
1 (6 years)
2 (7 years)
3 (8 years)
4 (9 years)
5 (10 years)
6 (11 years)
7 (12 years)
8 (13 years)
9 (14 years)
10 (15 years)
11 (16 years)
12 (17 years)
13 (18 years)
informal patterns and structures
patterns and simple number relations
representations (number line, table)
word problems
(anti-) proportional relationships (propaedeutic level)
(anti-) proportional relationships
representations (coordinate system)
linear functions
algebra
formal changes of representations
general properties of a function
quadratic functions
rational functions
transformations (optional)
linear function (with absolute value)
power functions
inverse functions
trigonometric functions
exponential functions
logarithmic functions
calculus
informal patterns and structures
patterns and simple number relations
representations
function machine
intuitive concept of a function through real life situations
representations
algebra (representations, use of algebraic expressions to represent relations, functions, proportional relationships)
word problems
(anti-) proportional relationships
formal changes of representations
concept of a function through correspondence
linear functions
general properties of a function
algebra
quadratic functions
trigonometric functions
general properties of a function
exponential functions
inverse functions
(basic) calculus
sequences
logarithmic functions

6. Interviews on functional thinking with educators

In this chapter, we summarize the results of our interviews about functional thinking with experts from the educational sector. The interviewees ranged from university staff for mathematics education to experienced primary school and secondary school mathematics teachers. The aim of these interviews was to gather the views on, and experiences with, functional thinking of professionals in mathematics education from primary to tertiary education. Between six and nine interviews were conducted in each partner country. The interviews were semi-structured with a focus on the person's understanding of functional thinking, ways to address functional thinking in class, and the implementation of the design principles introduced in Chapter 3. In the final part of the interview, the interviewees explained how well students achieve learning goals related to functional thinking and what students (further) need to learn about functional thinking. In addition, the interviewees named possible hinge points in students' learning process.

In our discussion of the results of these interviews, we focus on the understanding of functional thinking and on advice provided by the interviewees. This will be done for each country individually.

- **Cyprus:** Functional thinking was seen as relational thinking with a focus on the interdependence or relationship between variables. Functions were described as a rather strict mathematical concept, yet, the understanding of this concept was highlighted. Considerations related to patterns, sequences, and different representations were mentioned less often. The following points of advice were provided:
    - In primary education, pattern-based problems and real-life situations are often used to address functional thinking. Here, it is important to not only include recursive questions but to use far-generalization terms which require extracting the generalization rule.
    - In secondary education, students should describe what a function is in their own words based on their intuitive knowledge before formal notations are introduced. Further, students should be encouraged to identify the concept of functions in various situations including everyday problems, the use of technology, or word problems.
    - The understanding of relations is fundamental for understanding the concept of a function. A broader view can emphasize the connection to everyday life and other scientific domains. In addition, it is important to connect different representations at an early point in time.
- **Germany:** When asked to describe functional thinking, the mathematical relationship between two variables or quantities, as well as the concept of function were mentioned most often. Different representational forms and the change between these forms were also named frequently. The function aspects of correspondence, covariation, and mathematical object were mentioned in some interviews. The following advice was provided:
    - To work on an informal level as long as possible to support students' understanding.
    - The main focus should be on understanding and not on practicing procedures only. At the same time, it is important not to overemphasize one subtopic, e.g. proportionality.
    - The connection between functions and real-world, respectively current topics (e.g., dealing with functions in the context of the Corona crisis), should always be visible, also in higher grades. It is important to spark students' curiosity so they want to learn more.
    - Students should be able to understand the concept of function and be able to use it outside the classroom. This includes different forms of representation.

  - It is important to address functional thinking in a variety of ways, on a regular basis, as early in education as possible, even in less related topics, and as often as possible.
  - To support the covariational aspect in primary education.
- **Poland:** In Poland, the aspect of correspondence was mentioned in all interviews. In addition, the perception of the dependence between two variables by choosing one variable and the effect on the other one, functions as tools to solve problems and functions as mathematical objects were also stated often. The actions of generalization and ordering were named in some interviews. In addition to the description of functional thinking, the following advice was provided:
  - Functional thinking should be developed by different activities in different contexts with a focus on embodiment and activities for young learners. Moreover, the corresponding classroom should always include inquiry-based elements.
  - Students should discover a topic by analyzing examples from everyday life and with the help of discussions.
  - Basic concepts like variable should be emphasized first. In the further development students should observe phenomena and describe observations using mathematical language.
  - Functional thinking should not be treated as an isolated topic but in relation to other topics and subjects.
- **Netherlands:** When asked for a definition of functional thinking, interviewees from the Netherlands often named input-output procedures, domain and range, theoretical correspondence, actions on functions like differentiation and/or inverse functions. In addition, function as object, arrow scheme of input and output, guidelines for adequate (covariational) reasoning with functions appeared often within the interviews. Symbol sense, the use of different representations and the connection to real-world situations were mentioned in more than half of the interviews. Many interviewees also referred to the functionality (meaningfulness, applicability) of mathematics when asked for a definition, unrelated to the concept of function. The interviewees provided the following advice:
  - To focus on symbol sense and modeling, students need to comprehend meaningful mathematization.
  - Domain and range give insight in functions as processes; linked to the inverse it leads to reification (see Chapter 4 for explanation).
  - Establish clear guidelines for student reasoning within the (to be established) functional thinking framework. Use these guidelines to appraise the quality of students' functional reasoning.
- **Slovakia:** All interviewees in Slovakia mentioned the search for connections and relationships between values or variables when describing functional thinking. The relation between objects, dependencies between quantities or variables, as well as the connection to practical issues in society were also mentioned frequently. The following advice was provided:
  - Young students learn well through natural playfulness in groups. This should be used for teaching functional thinking.
  - To connect mathematics with everyday life. This does not only help for learning related to functional thinking but also beyond and can also be helpful to motivate students.
  - Engaging students to find more than one way to solve a problem and to think "out of the box" since this can visualize/indicate their learning process of functional thinking.
  - To teach more than just common procedures. Basic knowledge (e.g., variables, mathematical expressions, equations) is needed to apply the learned content.

This brief overview illustrates differences but also commonalities in the views on functional thinking expressed by the experts of the five partner countries. The idea of a connection between elements (variables, quantities, values, …) of two sets was mentioned in most interviews but in heterogeneous forms. Further descriptions and connotations mentioned by the experts were rather divergent. The provided advices emphasize the connection to real-world or everyday situations, as well as the focus on understanding and the importance of basic knowledge on variables and other topics as a basis for functional thinking. More detailed information of the interviews, as well as a comparative analysis of them will be provided on the project homepage.

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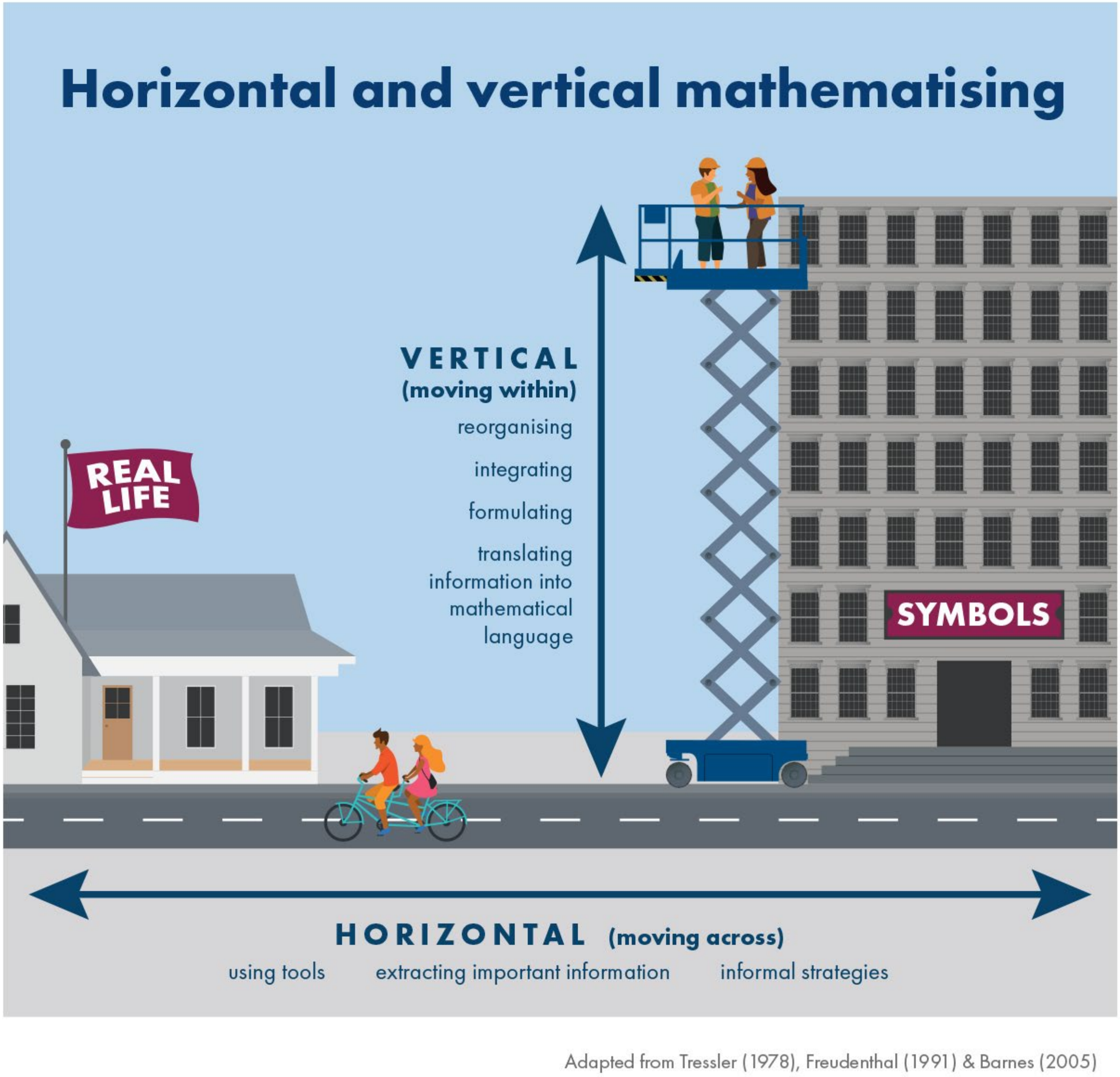
9. Appendices

9.1 Appendix A: An inquiry-based lesson plan

Target knowledge	A precise mathematical formulation of the goal.		
Broader goals	Broader achievements such as competences, possible applications, reasoning etc.		
Prerequisite mathematical knowledge	Precise formulation of what mathematical knowledge, skills and competences the students are expected to possess (before engaging with this situation).		
Grade	Grade number and age of students.		
Time	Estimated time and number of lessons. (45-60 minutes entities).		
Required material	All sorts of needed artefacts.		
Problem: The exact formulation of the main problem, which the teacher devolves to the students (possibly after some preparatory activity).			
	Teacher's actions incl. instructions	**Students' actions and reactions**	**Observations from implementation**
Handing the problem situation to the students Time estimate			
Students work on the problem situation Time estimate			
Students present their solutions Time estimate			
Students discuss whether their solutions are valid Time estimate			
Teacher connects the students' solution to the learning goals Time estimate			
Possible ways for students to realize target knowledge	- Be mathematically explicit about the strategies that students might follow. Remember to emphasize when a strategy can split into a scenario with ICT or without ICT using only pen and paper, as well as if the strategy requires to look at special cases.		
Further study	- What are possible applications / generalization of the notion or the concept studied?		
List of additional materials	- Students' productions (snapshots of boards, reports, assignments, posters etc.) - Formulations of students' assignments, reports or other productions required from students based on the lesson - Table for recording students' strategies - Video		

9.2 Appendix B: An illustration of horizontal and vertical mathematising



Source: cambridgemathematics.org

9.3 Appendix C: Literature review method

The literature search was conducted in four databases: ERIC, PsycINFO, Scopus, and Web of Science, and we searched for relevant studies published in peer-reviewed journals and written in English. There were no further methodological restrictions, so we included papers with qualitative studies, quantitative studies, and mixed-method studies. And we did not restrict the publication dates of the articles, because we are also interested in articles that did not (yet) mention embodied cognition as a primary or relevant theory, but still applied its core features, for example, in the field of kinesthetic learning. In the course of our ongoing search attempts, we defined a query consisting of Functional Thinking × (Embodiment OR Abstraction/Reification OR Digital Technology) (see Table 1 for the full query). Our initial search, conducted on Dec 7, 2020, yielded 397 journal articles. After deduplication, 333 unique publications remained.

Table 1 *Query and Filters*

Functional thinking	("Function* thinking" OR "Function* reasoning" OR "Function* relation" OR "math* Function* " OR "covariation* reasoning" OR "Function* approach" OR "thinking function*")
	AND
Embodiment Dub	(embod* OR enactment OR sensorimotor OR kines* OR perception OR action-perception OR "body motion" OR "physical experience" OR "physical participa*")
	OR
Abstraction/Reification	(abstracti* OR reification OR "math* abstract*" OR encapsulation OR "object formation" OR "concept imag*" OR visualization)
	OR
Digital technology	("digital technolog*" OR "digital tool*" OR "physical tool*" OR "ICT tool*" OR ICT OR GeoGebra)
	AND
Domain	(math* OR "math* education" OR "math* instruction" OR "physical science" OR science OR stem OR "teaching method" OR education* OR learning)
	AND
Filter(s)	• English language • In SCOPUS and Web of Science, the limitations were set to journal articles and conference proceedings. • In ERIC, the limitations were set to journal articles and peer-reviewed articles.

The literature selection round started with the detailed information, such as title, abstract and keywords of the article, to judge each article's relevance to each of the four aspects:

- Functional thinking (FT): This is the general dimension: does the paper address functional thinking, functional reasoning, covariational reasoning,…?
- Embodiment (EM): This is the specific aspect of embodiment in discussing functional thinking: does the paper address embodiment, enactivism, bodily enactment, movement, …?
- Abstraction (AB): Does the paper address abstraction in discussing functional thinking, object formation, reification, encapsulation, procepts, …?

- Digital Technology (DT): Does the paper address the use of digital technology in the teaching and learning of functional thinking?

At the end of this round, 177 papers – empirical as well as theoretical papers - were initially collected (Figure 1) with the help of ten coders.

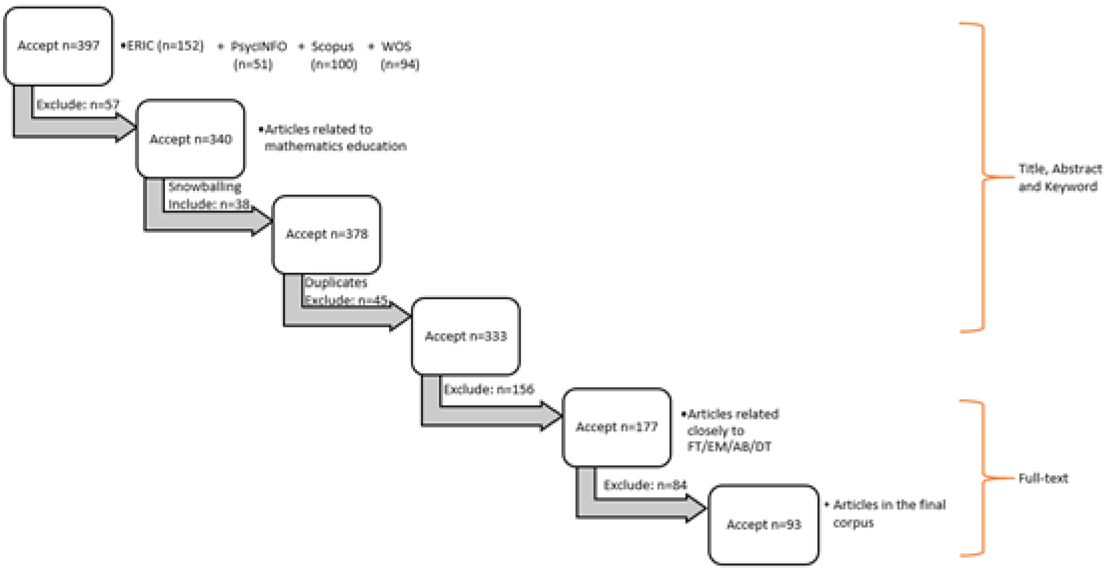


Figure 1 Flowchart of search strategy showing the numbers of included and excluded articles

The categories of the selected papers (n=177) with regard to the aspects and education levels are shown in Figure 2. The three numbers in each category correspond to primary education, secondary education, and tertiary education, respectively.

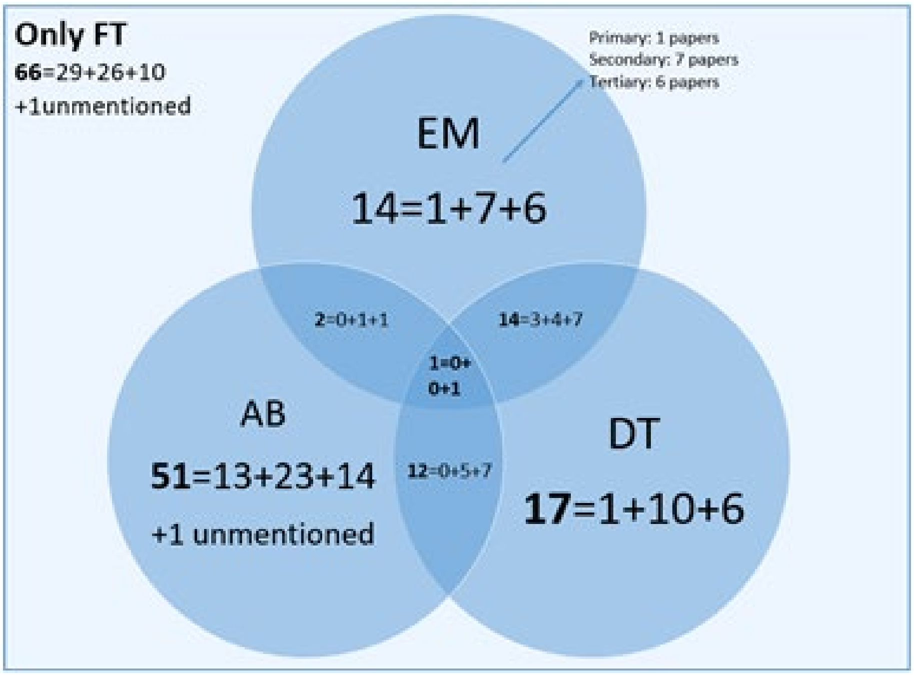


Figure 2 The Venn diagram of categories

After that, fifteen coders participated in the literature appraisal round, during which each coder read full texts and finished a spreadsheet that contains the core ideas of each article. The following four core questions respond to four foci of our project：

- FT: Which definitions / views on FT are used, including understanding of function?
- EM: How do action/perception loops, sensorimotor experiences, and gestures contribute to FT?

- DT: How does the tool use contribute to which aspects of functional thinking?
- AB: What does the paper say about abstraction/reification in the framework of functional thinking?

This resulted in the exclusion of eighty-four articles and the final selection of ninety-three articles for our review. We directly removed the articles coded 0 to 2 as they are perceived as less helpful to our project (n=43). And then, according to the structure of the document, we integrated core ideas from each article regarding their theoretical contribution, design contribution and implementation contribution. As a result, ninety-three articles are included in the final corpus. The information from those articles supports to identify building blocks for this Literature Review section in the vision document. The following Venn diagram presents the categories of the final corpus.

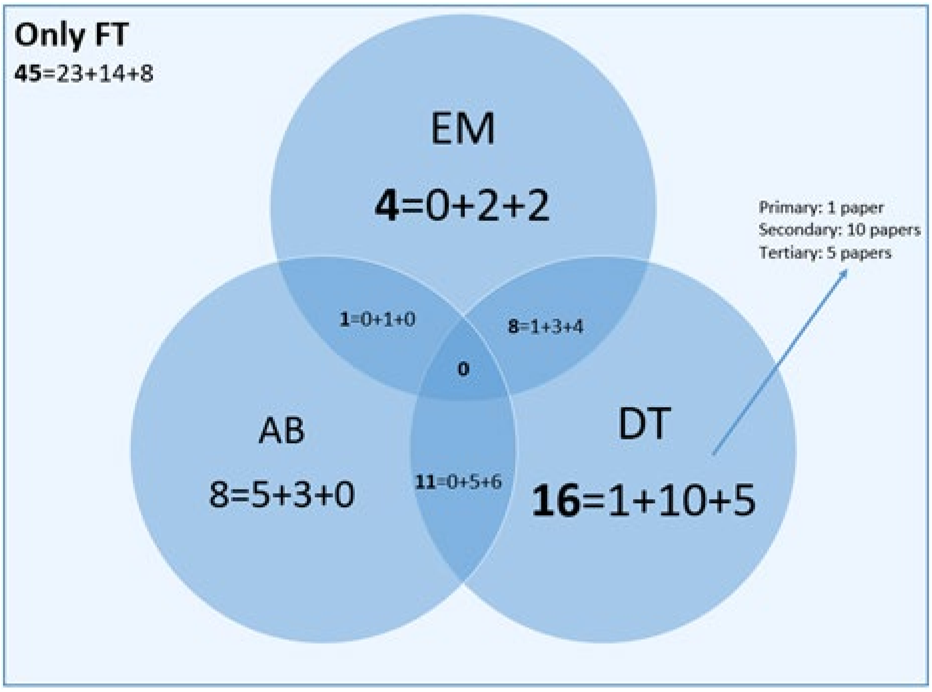


Figure 3 The Venn diagram of final categories